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Editorial

We start this pioneering work, which do not seek perfection as much as aiming to provide a scientific window that opens a wide area for all the distinctive pens, both in the University of Sabratha or in other universities and research centers. This emerging scientific journal seeks to be a strong link to publish and disseminate the contributions of researchers and specialists in the fields of applied science from the results of their scientific research, to find their way to every interested reader, to share ideas, and to refine the hidden scientific talent, which is rich in educational institutions. No wonder that science is found only to be disseminated, to be heard, to be understood clearly in every time and place, and to extend the benefits of its applications to all, which is the main role of the University and its scholars and specialists. In this regard, the idea of issuing this scientific journal was the publication of the results of scientific research in the fields of applied science from medicine, engineering and basic sciences, and to be another building block of Sabratha University, which is distinguished among its peers from the old universities.

As the first issue of this journal, which is marked by the Journal of Applied Science, the editorial board considered it to be distinguished in content, format, text and appearance, in a manner worthy of all the level of its distinguished authors and readers.

In conclusion, we would like to thank all those who contributed to bring out this effort to the public. Those who lit a candle in the way of science which is paved by humans since the dawn of creation with their ambitions, sacrifices and struggle in order to reach the truth transmitted by God in the universe. Hence, no other means for the humankind to reach any goals except through research, inquiry, reasoning and comparison.

Editorial Committee

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All submitted research manuscripts must follow the following pattern:

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- Author Name, Affiliation and Email
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- Introduction.
- Methodology.
- Results and Discussion.
- Conclusion.
- Acknowledgments (optional).
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Invitation

The Editorial Committee invites all researchers "Lectures, Students, Engineers at Industrial Fields" to submit their research work to be published in the Journal. The main fields targeted by the Journal are:

- Basic Science.
- Medical Science & Technology.
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CONTENTS

SOLUTION OF ABEL'S INTEGRAL EQUATION USING ABAOUB-SHKHEAM TRANSFORM

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Abstract

In this paper, the authors determine the exact solution of a special case of Volterra integral equations of the first kind when the kernel k(x,t)=(x−t)−αk(x,t) = (x-t)^{-\alpha}k(x,t)=(x−t)−α is not defined at certain values within its domain. Using the Abaoub-Shkheam transform, this equation, known as Abel's integral equation, is specifically studied for α=1/2\alpha = 1/2α=1/2. Several numerical problems have been analyzed and solved with the assistance of the Abaoub-Shkheam transform. The results of these numerical problems demonstrate that the Abaoub-Shkheam transform is a highly effective integral transform for determining the exact solution of this equation.

Keywords: Abel's integral equation; generalized Abel's integral equation; Abaoub-Shkheam transforms.

Introduction

Abel's integral equation is one of the very first integral equations to be seriously studied. Abel was led to this equation by a mechanics problem involving the motion of a particle on a smooth curve lying in a vertical plane. However, his equation has since found applications in diverse fields, such as atomic scattering, mechanics, radio astronomy, physics, electron emission, X-ray radiography, and seismology. More specifically, in mathematics, Abel's integral equation is pertinent to certain inverse problems in partial differential equations and special problems in the theory of Brownian motion (Aggarwal et al., 2018). Abel later generalized his original problem by introducing the singular integral equation:

$$\boldsymbol{f(x) = \int_0^x (x-t)^{-\alpha}\, g(t)\, dt, \qquad 0 < \alpha < 1} \quad (1)$$

known as the Generalized Abel's integral equation where α are known constants such that $0 < \alpha < 1, f(x)$ is a predetermined data function, and $g(x)$ is the solution that will be determined. When $\alpha = {}^{1}/_{2}$, is a special case of the generalized Equation (1).

It is to be noted that Abel's integral equation (1) is also called Volterra integral equation of the first kind. Besides the kernel $K(x, t)$ in Abel's integral Equation (1) is:

$$\boldsymbol{k(x,t) = (x-t)^{-\alpha}}$$

where $k(x,t)$ tends to infinity as t tends to x (w).

Integral transforms are widely used mathematical techniques because they provide exact solutions to problems without requiring extensive calculations. Various researchers have applied different integral transformations to solve Volterra integral equations. For instance, Aggarwal et al. (2018), Aggarwal et al. (2019), and Aggarwal et al. (2018) analyzed and determined analytical solutions for first- and second-kind Volterra integral equations. Similarly, Shehu, Elzaki, and Ezaki (2011), as well as Mohand, Aboodh, and Abdelbagy (2016), contributed to solutions of Volterra integro-differential equations of the first kind.

Abaoub and Shkheam (A. Abaoub and A. Shkheam(2020)) defined “Q- transform'' of the function (t) for all $t \geq 0$ in the year 2020 as

$$\boldsymbol{Q[f] = \int_0^\infty f(ut)\, e^{\frac{-t}{s}}\, dt = T(u,s)}$$

where the operator Q is called the Abaoub-Shkheam transform operator.

In this paper, we are presenting a new application of Abaoub-Shkheam transform for solving Abel's integral equation and explain all procedures by giving some numerical applications

Some Useful Properties of Q- Transform

1 Linearity property of Q- transforms (A. Abaoub and A. Shkheam, 2020)**:**

If Q- transform of functions $F_1(t)$ and $F_2(t)$ are $T_1(u,s)$ and $T_2(u,s)$ respectively then Q- transform of $af_1(t) + bf_2(t)$ is given by $(aT_1(t) + bT_2(t))$, where a,b are arbitrary constants

2 Q- transform of the derivatives of the function $\boldsymbol{(t)}$:

Let $Q\{f(t)\} = T(u.s)$, then

a) $\boldsymbol{Q\{F'(t)\} = \frac{1}{us}T(u,s) - \frac{1}{u}F(0)}$

b) $\boldsymbol{Q\{F''(t)\} = \frac{1}{u^2s^2}T(u,s) - \frac{1}{u^2s}F(0) - \frac{1}{u}F'(0)}$

c) $\boldsymbol{Q\{F^{(n)}(t)\} = \frac{1}{u^n s^n} T(u,s) - \frac{1}{u} \sum_{k=0}^{n-1} \frac{F^{(K)}(0)}{(us)^{u-k-1}} (0)}$

3 Convolution theorem for Q- transforms:

If Q- transform of functions $F_1(t)$ and $F_2(t)$ are $T_1(u,t)$ and $T_2(u,t)$ respectively then Q- transform of their convolution $F_1(t) * F_1(t)$is given by

$$\boldsymbol{Q\{F_1(t) * F_2(t)\} = uQ\{F_1(t)\} * Q\{F_2(t)\} = uT_1(u,t) * T_2(u,t)}$$

where

$F_1(t) * F_2(t)$ is defined by $F_1(t) * F_1(t) = \int_0^t F_1(t-x) F_2(x) dt = \int_0^t F_1(x) F_2(t-x) dt$**.**

$Proof: see$ (A. Abaoub, and A. Shkheam, 2020).

The Generalized Abel's Integral Equation

In this section, Q- transform is going to be used to find the exact solution of Abel's integral equation. Abel's integral equation is defined as,

$$\boldsymbol{f(x) = \int_0^x \frac{1}{(x-t)^\alpha} g(t) dt \qquad ; 0 \leq \alpha \leq 1} \tag{13}$$

To determine a formula that will be used for solving the generalized Abel's integral equation (1), we will apply the Q- transform method in a parallel manner to the approach followed before. Taking Q- transforms of both sides of (13) leads to

$$\boldsymbol{Q[f(x)] = Q\left[\int_0^x \frac{1}{(x-t)^\alpha} g(t) dt\right]} \tag{14}$$

$$\boldsymbol{Q[f(x)] = uQ[x^\alpha] Q[g(t)]}$$

or equivalently

$$\boldsymbol{T_1(u,s) = u \frac{\Gamma(1-\alpha)}{u^\alpha} s^{1-\alpha} T_2(u,s)} \tag{15}$$

that gives

$$\boldsymbol{T_2(u,s) = \frac{1}{u^{1-\alpha} s^{1-\alpha} \Gamma(1-\alpha)} T_1(u,s)} \tag{16}$$

where Γ is the gamma function. The last equation (16) can be rewritten as

$$\boldsymbol{Q[g(x)] = \frac{u u^{\alpha-1} s^\alpha \Gamma(\alpha)}{us \Gamma(\alpha) \Gamma(1-\alpha)} Q[f(x)]}$$

$$\boldsymbol{Q}[\boldsymbol{g}(\boldsymbol{x})] = \frac{\mathbf{1}}{\boldsymbol{us}\boldsymbol{\Gamma}(\boldsymbol{\alpha})\boldsymbol{\Gamma}(\mathbf{1}-\boldsymbol{\alpha})}\boldsymbol{Q}[\boldsymbol{f}(\boldsymbol{x})] \qquad \textbf{(17)}$$

Where

$$\boldsymbol{Q}[\boldsymbol{f}(\boldsymbol{x})] = \int_{\mathbf{0}}^{\boldsymbol{x}} \frac{\mathbf{1}}{(\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{\alpha}-\mathbf{1}}}\boldsymbol{f}(\boldsymbol{t})\boldsymbol{dt} \qquad \textbf{(18)}$$

Using the facts

$$\mathbf{Q}[\boldsymbol{f}'(\boldsymbol{x})] = \frac{\mathbf{1}}{\boldsymbol{us}}\boldsymbol{Q}[\boldsymbol{F}(\boldsymbol{x})] - \frac{\mathbf{1}}{\boldsymbol{u}}\boldsymbol{F}(\mathbf{0}) \qquad \textbf{(19)}$$

And

$$\boldsymbol{\Gamma}(\boldsymbol{\alpha})\boldsymbol{\Gamma}(\mathbf{1}-\boldsymbol{\alpha}) = \frac{\boldsymbol{\pi}}{\sin \boldsymbol{\alpha\pi}} \qquad \textbf{(20)}$$

into (17) we obtain

$$\boldsymbol{Q}[\boldsymbol{g}(\boldsymbol{x})] = \frac{\sin \boldsymbol{\alpha\pi}}{\boldsymbol{\pi}}\boldsymbol{Q}[\boldsymbol{f}'(\boldsymbol{x})] \qquad \textbf{(21)}$$

Applying Q^{-1} to both sides of (21) gives the formula

$$\boldsymbol{g}(\boldsymbol{x}) = \frac{\sin \boldsymbol{\alpha\pi}}{\boldsymbol{\pi}}\frac{\boldsymbol{d}}{\boldsymbol{dx}}\int_{\mathbf{0}}^{\boldsymbol{x}} \frac{\mathbf{1}}{(\boldsymbol{x}-\boldsymbol{t})^{\mathbf{1}-\boldsymbol{\alpha}}}\boldsymbol{f}(\boldsymbol{t})\boldsymbol{dt} \qquad \textbf{(22)}$$

Integrating the integral at the right side of (22) and differentiating the result we obtain the more suitable formula

$$\boldsymbol{g}(\boldsymbol{x}) = \frac{\sin \boldsymbol{\alpha\pi}}{\boldsymbol{\pi}}\left[\frac{\boldsymbol{f}(\mathbf{0})}{\boldsymbol{x}^{\mathbf{1}-\boldsymbol{\alpha}}}\right] + \int_{\mathbf{0}}^{\boldsymbol{x}} \frac{\boldsymbol{f}'(\boldsymbol{t})}{(\boldsymbol{x}-\boldsymbol{t})^{\mathbf{1}-\boldsymbol{\alpha}}}\boldsymbol{dt} \qquad ;\mathbf{0} \leq \boldsymbol{\alpha} \leq \mathbf{1} \qquad \textbf{(23)}$$

When $\alpha = 1/2$, is a special case of the generalized equation (1). Using Q- transform we get

$$\boldsymbol{g}(\boldsymbol{x}) = \frac{\mathbf{1}}{\boldsymbol{\pi}}\frac{\boldsymbol{d}}{\boldsymbol{dx}}\int_{\mathbf{0}}^{\boldsymbol{x}} \frac{\mathbf{1}}{(\boldsymbol{x}-\boldsymbol{t})^{\boldsymbol{\alpha}}}\boldsymbol{f}(\boldsymbol{t})\boldsymbol{dt} \qquad \textbf{(24)}$$

Four remarks can be made here:

1. The kernel is called weakly singular as the singularity may be transformed away by a change of variable.
2. The exponent of the kernel of the generalized Abel's integral equation is $-\alpha$, but the exponent of the kernel of the two formulae (22) and (23) is $\alpha-1$.
3. The unknown function in (13) has been replaced by $f(t)$ and $f'(t)$ in (22) and (23) respectively.

4. 4.In (22), differentiation is used after integrating the integral at the right side, whereas in (23), integration is only required.

Applications

In this section, we present some numerical applications to demonstrate the effectiveness of Q- transform to solve Abel's integral equation, by Using formula (12) to determine the solution of Abel's problem (2)

Example 1: Consider the Abel's integral equation:

$$\pi x = \int_0^x \frac{u(t)}{(x-t)^{\frac{2}{3}}} dt$$

Using (22) given

$$u(x) = \frac{\sin\frac{2}{3}\pi}{\pi} \int_0^x \frac{f(x)}{(x-t)^{\frac{1}{3}}} dt$$

$$= \frac{\sqrt{3}}{2} \frac{d}{dx} \int_0^x \frac{t}{(x-t)^{\frac{1}{3}}} dt = \frac{3\sqrt{3}}{4} x^{\frac{2}{3}}$$

Example 2: Consider the Abel's integral equation:

$$\sqrt{x} + \frac{8}{3} x^{\frac{3}{2}} = \int_0^x \frac{1}{\sqrt{x-t}} g(t) dt$$

Using (12) gives

$$g(x) = \frac{1}{\pi} \left[\frac{d}{dx} \int_0^x \frac{\sqrt{t} + \frac{8}{3} t^{\frac{3}{2}}}{\sqrt{x-t}} dt \right] = 1 + 2x$$

Conclusion

In this paper, Abaoub-Shkheam transform has been used to find the exact solution for the Abel's integral equation. It has been concluded from solving some Abel's problems using Abaoub-Shkheam transform that Abaoub-Shkheam transform is successful tool which gives the same solution given by using other integral transform easily.

References

- Abaoub, A. and Shkheam, A. (2020). The New Integral Transform "Abaoub-Shkheam Transform". *IAETSD Journal for Advanced Research in Applied Sciences*.

- Abaoub, A. and Shkheam, A. (2020). Utilization Abaoub-Shkheam Transform in Solving Linear Integral Equations of Volterra. *International Journal of Software & Hardware Research in Engineering (IJSHRE)*, 8(12).

- Abood, K. (2014). Application of new transform "Aboodh Transform" to partial differential equations. *Global Journal of Pure and Applied Mathematics*, 10(2), pp. 249–254.

- Abdelilah, K. and Hassan, S. (2017). The use of Kamal transform for solving partial differential equations. *Advances in Theoretical and Applied Mathematics*, 12(1), pp. 7–13.

- Aggarwal, S., Asthana, N. and Singh, D. (2019). Solution of population growth and decay problems by using Aboodh transform method. *International Journal of Research in Advent Technology*, 6(10), pp. 2706–2710.

- Aggarwal, S., Chauhan, R. and Sharma, N. (2018). Application of Aboodh transform for solving linear Volterra integro-differential equations of the second kind. *International Journal of Research in Advent Technology*, 6(6), pp. 1186–1190.

- Aggarwal, S., Chauhan, R. and Sharma, N. (2018). Application of Elzaki transform for solving linear Volterra integral equations of the first kind. *International Journal of Research in Advent Technology*, 6(12), pp. 3687–3692.

- Aggarwal, S., Chauhan, R. and Sharma, N. (2018). Mohand transform of Bessel's functions. *International Journal of Research in Advent Technology*, 6(11), pp. 3034–3038.

- Aggarwal, S., Chauhan, R. and Sharma, N. (2018). New application of Kamal transform for solving linear Volterra integral equations. *Journal of Latest Technology in Engineering, Management & Applied Science*, 7(4), pp. 138–140.

- Aggarwal, S. and Chaudhary, R. (2019). A comparative study of Mohand and Laplace transforms. *Journal of Emerging Technologies and Innovative Research*, 6(2), pp. 230–240.

- Aggarwal, S. and Gupta, A. (2019). Solution of linear Volterra integro-differential equations of the second kind using Kamal transform. *Journal of Emerging Technologies and Innovative Research*, 6(1), pp. 741–747.

- Aggarwal, S. and Gupta, A. (2019). Sumudu transform for the solution of Abel's integral equation. *Journal of Emerging Technologies and Innovative Research*, 6(4), pp. 423–431.

- Aggarwal, S., Pandey, M., Asthana, N., Singh, D. and Kumar, A. (2018). Application of Mahgoub transform for solving population growth and decay problems. *Journal of Computer and Mathematical Sciences*, 9(10), pp. 1490–1496.

- Aggarwal, S., Sharma, S. and Gupta, A. (2019). A new application of Mohand transform for handling Abel's integral equation. *Journal of Emerging Technologies and Innovative Research*, 6(3), pp. 600–608.

- Aggarwal, S. and Sharma, S. (2019). A comparative study of Mohand and Sumudu transforms. *Journal of Emerging Technologies and Innovative Research*, 6(3), pp. 145–153.

- Aggarwal, S. and Sharma, S. (2019). Solution of Abel's integral equation by Aboodh transform method. *Journal of Emerging Technologies and Innovative Research*, 6(4), pp. 317–325.

- Aggarwal, S., Sharma, N. and Chauhan, R. (2018). Application of Aboodh transform for solving linear Volterra integral equations of the first kind. *International Journal of Research in Advent Technology*, 6(12), pp. 3745–3753.

- Aggarwal, S., Sharma, N. and Chauhan, R. (2018). Solution of population growth and decay problems by using Mohand transform. *International Journal of Research in Advent Technology*, 6(11), pp. 3277–3282.

- Belgacem, F. and Karaballi, A. (2006). Sumudu transform: Fundamental properties investigations and applications. *Journal of Applied Mathematics and Stochastic Analysis*, pp. 1–23.

- Chauhan, R. and Aggarwal, S. (2018). Solution of linear partial integro-differential equations using Mahgoub transform. *Periodic Research*, 7(1), pp. 28–31.

- Debnath, L. and Bhatta, D. (2006). *Integral transforms and their applications*. 2nd edn. Chapman & Hall/CRC.

- Elzaki, T. and Ezaki, S. (2011). Applications of new transform "Elzaki transform" to partial differential equations. *Global Journal of Pure and Applied Mathematics*, 7(1), pp. 65–70.

- Elzaki, T. and Ezaki, S. (2011). On the Elzaki transform and ordinary differential equation with variable coefficients. *Advances in Theoretical and Applied Mathematics*, 6(1), pp. 41–46.

- Kumar, P., Gomathi, P. and Viswanathan, A. (2018). Applications of Mohand transform to mechanics and electrical circuit problems. *International Journal of Research in Advent Technology*, 6(10), pp. 2838–2840.

- Kumar, P., Gnanavel, M. and Viswanathan, A. (2018). Application of Mohand transform for solving linear Volterra integro-differential equations. *International Journal of Research in Advent Technology*, 6(10), pp. 2554–2556.

- Mohand, D., Aboodh, K. and Abdelbagy, A. (2016). On the solution of ordinary differential equations with variable coefficients using Aboodh transform. *Advances in Theoretical and Applied Mathematics*, 11(4), pp. 383–389.

- Rahman, M. (2007). *Integral equation and their applications*. Wit Press, Southampton, Boston.

- Wazwaz, A. (2011). *Linear and nonlinear integral equations: Methods and applications*. Higher Education Press, Beijing.

- Watugula, G. (1993). Sumudu transform: A new integral transform to solve differential equations and control engineering problems. *International Journal of Mathematical Education in Science and Technology*, 24(1), pp. 35–43.