Sharp Convergence of the Green's Function–Homotopy Perturbation Method and Detection of Turning Points in Nonlinear Boundary Value Problems

Maha Al-Sharif[1], Mohaned Alkbuer[2]

[1]Department of General Sciences, Faculty of Engineering, University of Sabratha, Sabratha, Libya

[2]Department of General Sciences, Faculty of Engineering, University of Sabratha, Sabratha, Libya

Corresponding Author: E-mail: maha.alshareef@sabu.edu.ly

Abstract:

The homotopy perturbation method applied to the Green's-function (Hammerstein) form of a nonlinear two-point boundary value problem produces a series whose terms satisfy the boundary conditions and are obtained by single integrations. This coupling is known, and for the homotopy considered it coincides with the Adomian decomposition method in integral form. The present paper studies when, and to what, the resulting series converges. Three results of increasing sharpness are proved: a contraction theorem with a geometric error bound; a majorant theorem that reduces convergence to a scalar inequality and gives closed-form coefficient bounds of Catalan and Lambert-W type; and, when the kernel, the source and the nonlinearity have fixed signs, a Pringsheim-type theorem showing that the series converges as far as the analytic solution branch extends and diverges beyond a fold of that branch. For the Bratu problem this yields convergence if and only if $0 < \lambda \leq \lambda_c = 8(\alpha^2 - 1) \approx 3.513831$, where $\alpha \tanh \alpha = 1$, whereas the first two results guarantee only $\lambda < 2.324$ and $\lambda \leq 8/e$. For a Fredholm equation with the kernel $e^{-|x-t|}$, the linearisation at the intended solution becomes singular at $\lambda_t = (1 + k_1^2)/4 \approx 0.676763$; numerical branch tracking indicates that beyond this value the series converges to a different solution. The computed coefficients are consistent with the decay $n^{-3/2}\mu_c^{-n}$ expected at a quadratic fold, and extrapolation of 60 terms reproduces λ_c to seven significant digits. Comparisons with Picard iteration, finite differences and the Nyström method indicate that the approach is most useful in symbolic and parametric computations.

Keywords: Green's function; homotopy perturbation method; Bratu problem; turning point; majorant series.

1. Introduction

Nonlinear integral equations and nonlinear boundary value problems (BVPs) arise throughout mathematical physics and engineering, and closed-form solutions are rarely available. Among the semi-analytical techniques developed for such problems, the homotopy perturbation method (HPM) introduced by He (1999, 2000) has been used widely. HPM embeds the problem in a one-parameter family (a homotopy) that deforms an easily solvable problem at $p = 0$ into the original problem at $p = 1$, and expands the solution as a power series in the embedding parameter $p \in [0,1]$. Unlike classical perturbation methods, it does not require a small physical parameter in the original equation (He, 1999). HPM has been applied successfully to linear and nonlinear integral equations (Abbasbandy, 2006; Ghasemi et al., 2007), and its convergence has been analysed in several settings (Biazar & Ghazvini, 2009; Ayati & Biazar, 2015).

When HPM is applied directly to a BVP in differential form, each correction term is obtained by solving a linear differential equation, and the boundary conditions must be imposed by hand at every order to determine the constants of integration. The Green's function removes this burden: it inverts the linear part of the operator together with the boundary conditions, so the BVP becomes an equivalent Fredholm integral equation whose kernel already encodes the boundary data (Stakgold & Holst, 2011).

Related work. Combining Green's functions with series and iterative methods is an established idea. Wang et al. (2009) proposed exactly the coupling studied here for second-order two-point BVPs: the Green's function first transforms the differential equation into an equivalent integral equation that already satisfies the boundary conditions, and HPM is then applied to that integral equation; their work demonstrates the accuracy of the algorithm on numerical examples. Singh et al. (2015) applied the Adomian decomposition method (ADM) to the Green's-function form of two-point BVPs. Singh and Wazwaz (2016) did the same for fourth-order integro-differential BVPs and proved convergence under a Lipschitz condition. Sawangtong

and Sawangtong (2019) proposed a Green's function homotopy perturbation method for linear initial-boundary value problems. Their convergence theorem for the series assumes a ratio condition $\|v_n\| \leq \sigma\|v_{n-1}\|$ on the computed terms. Khuri and Sayfy (2014) interpreted the variational iteration method (VIM) as a fixed-point iteration built on the Green's function, and Khuri and Louhichi (2018) embedded Green's functions in an Ishikawa-type scheme. The convergence of ADM itself has been studied since the work of Cherruault (1989) and Abbaoui and Cherruault (1994), including for integral equations (Badredine et al., 1999). ADM has also been applied to the Bratu problem (Wazwaz, 2005), and Kafri and Khuri (2016) solved the Bratu problem by fixed-point iterations built on its Green's function. More recently, Jyoti and Singh (2022) developed an HPM-based iterative technique for Bratu-type problems, and Tomar et al. (2023) proposed an iterative method based on Green's functions for nonlinear two-point BVPs, including the Bratu problem, together with a convergence analysis. For initial-value problems, Abdelrazec and Pelinovsky (2011) proved convergence of ADM with the method of majorants of Cauchy–Kovalevskaya type. Al-Hayani and Casasús (2005) applied ADM to parameter-dependent BVPs that exhibit turning points. Only the abstract of that work was available to us, so we cannot say whether their notion of a turning point coincides with the fold of a solution branch studied here; in singular perturbation theory the term usually refers to a zero of a coefficient of the equation. We are not aware of results that determine the exact convergence region of decomposition-type series for boundary value problems, or that relate it to the turning points of the solution branch.

Contributions. We do not claim the coupling itself as new: it is the algorithm of Wang et al. (2009), and for the homotopy used here HPM–Green produces the same terms as ADM in Green's-function form (Remark 2), so all results below apply to both methods. Our contributions concern the convergence of the series:

(i) an *a priori* contraction theorem (Theorem 1) with an explicit ball-invariance condition and a geometric error bound;

(ii) a *majorant theorem* (Theorem 2) that reduces convergence to a scalar inequality and gives closed-form bounds of Catalan and Lambert-W type, which enlarge the guaranteed ranges substantially (Table (5));

(iii) a *sharp convergence theorem* (Theorem 3): under sign conditions, the radius of convergence in the embedding parameter is at least the extent of the analytic solution branch, and equals it when the branch ends at a fold. For the Bratu problem this is proved to give convergence if and only if $\lambda \leq \lambda_c$;

(iv) for Example 3, the closed-form value λ_t at which the linearisation at the intended solution becomes singular, together with numerical evidence that the series converges to a different solution beyond λ_t;

(v) a procedure for estimating turning points from the HPM terms by Domb–Sykes extrapolation, whose basis (the coefficient asymptotics) is supported numerically rather than proved (Section 3.4);

(vi) a comparison with Picard iteration, finite differences and the Nyström method (Section 3.5).

2. Methodology

2.1 Green's function and the integral reformulation

Consider the nonlinear two-point BVP

$$Ly + \mathcal{N}[y] = f(x), \qquad a < x < b, \tag{1}$$

subject to homogeneous separated boundary conditions

$$B_a[y] := \alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad B_b[y] := \beta_1 y(b) + \beta_2 y'(b) = 0. \tag{2}$$

Here L is a linear second-order differential operator, typically of Sturm–Liouville type

$$Ly = (p(x)\, y')' + q(x)\, y, \qquad p > 0 \text{ on } [a, b], \tag{3}$$

and $\mathcal{N}$ is a nonlinear term. Inhomogeneous boundary data can be reduced to (2) by subtracting a smooth function that satisfies them.

Definition 1 (Green's function (Stakgold & Holst, 2011)). Assume that the homogeneous problem $Lu = 0$ with (2) has only the trivial solution. The Green's function $G(x,\xi)$ of L with the boundary conditions (2) is the solution of $LG(\cdot,\xi) = \delta(\cdot - \xi)$ that satisfies the boundary conditions. Equivalently, it has the following properties:

(G1) $G(\cdot,\xi)$ satisfies $LG = 0$ for $x \neq \xi$ and satisfies (2);

(G2) G is continuous at $x = \xi$;

(G3) $\partial_x G$ has a jump of magnitude $1/p(\xi)$ at $x = \xi$: $\partial_x G(\xi^+,\xi) - \partial_x G(\xi^-,\xi) = 1/p(\xi)$;

(G4) if L with (2) is self-adjoint, as is the case for (3), then $G(x,\xi) = G(\xi,x)$.

If u_1 satisfies $Lu_1 = 0$ and the left condition, and u_2 satisfies $Lu_2 = 0$ and the right condition, then

$$G(x,\xi) = \frac{u_1(x_<)\,u_2(x_>)}{p(\xi)\,W(\xi)}, \qquad x_< = \min(x,\xi),\ x_> = \max(x,\xi), \tag{4}$$

where $W = u_1 u_2{}' - u_1{}' u_2$ and $p\,W$ is constant by Abel's identity (Stakgold & Holst, 2011).

Moving the nonlinear term to the right-hand side, put $\widetilde{\mathcal{N}} := -\mathcal{N}$. Then $Ly = f + \widetilde{\mathcal{N}}[y]$, and inverting L with the Green's function gives

$$y(x) = F(x) + \int_a^b G\,(x,\xi)\,\widetilde{\mathcal{N}}[y(\xi)]\,\mathrm{d}\xi, \qquad F(x) := \int_a^b G\,(x,\xi)\,f(\xi)\,\mathrm{d}\xi. \tag{5}$$

Equation (5) is a nonlinear Fredholm integral equation of the second kind (of Hammerstein type) whose kernel is the Green's function. Any continuous solution of (5) is a classical solution of (1)–(2), and conversely. We write K for the linear integral operator $(Ku)(x) = \int_a^b G\,(x,\xi)u(\xi)\,\mathrm{d}\xi$, so that (5) reads $y = F + K\widetilde{\mathcal{N}}[y]$. On $C[a,b]$ with the supremum norm,

$$\|K\| \le \gamma := \max_{x\in[a,b]} \int_a^b |G(x,\xi)|\,\mathrm{d}\xi. \tag{6}$$

Lemma 1. For $L = \mathrm{d}^2/\mathrm{d}x^2$ on $[0,1]$ with $y(0) = y(1) = 0$,

$$G(x,\xi) = \begin{cases} \xi\,(x-1), & 0 \le \xi \le x, \\ x\,(\xi-1), & x \le \xi \le 1, \end{cases} \qquad \int_0^1 |G(x,\xi)|\,\mathrm{d}\xi = \frac{x(1-x)}{2}, \qquad \gamma = \frac{1}{8}. \tag{7}$$

Moreover, Ku is the unique solution of $w'' = u$, $w(0) = w(1) = 0$.

Proof. With $u_1 = x$, $u_2 = x - 1$, $p \equiv 1$ we have $W = 1$, and (4) gives (7). Since $G \le 0$, $\int_0^1 |G|\,\mathrm{d}\xi = (1-x)\int_0^x \xi\,\mathrm{d}\xi + x\int_x^1 (1-\xi)\,\mathrm{d}\xi = x(1-x)/2$, whose maximum $1/8$ is attained at $x = 1/2$. The last statement is the defining property of G. ▫

2.2 The homotopy perturbation method for the integral equation

Following He (1999, 2000), we embed (5) into the family

$$\mathcal{H}(v,p): \qquad v(x) = F(x) + p\int_a^b G\,(x,\xi)\,\widetilde{\mathcal{N}}[v(\xi)]\,\mathrm{d}\xi, \qquad p \in [0,1]. \tag{8}$$

At $p = 0$ this gives $v = F$, the solution of the linear problem $Ly = f$ with (2); at $p = 1$ it is the original equation (5). We seek

$$v = v_0 + p\,v_1 + p^2 v_2 + \cdots, \qquad y = \lim_{p\to 1} v = \textstyle\sum_{n\ge 0} v_n, \tag{9}$$

and expand the nonlinearity as $\widetilde{\mathcal{N}}[v] = \sum_{n\ge 0} p^n\,A_n$, where

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}p^n}\,\widetilde{\mathcal{N}}\Big[\textstyle\sum_{i\ge 0} p^i\,v_i\Big]\Big|_{p=0}, \qquad n \ge 0, \tag{10}$$

are the Adomian polynomials (Rach, 1984; Adomian, 1994). Note that A_n depends only on $v_0,\dots,v_n$. Substituting into (8) and equating coefficients of equal powers of p gives the recursion

$$v_0(x) = F(x), \qquad v_{n+1}(x) = \int_a^b G\,(x,\xi)\,A_n(\xi)\,\mathrm{d}\xi, \qquad n \ge 0. \tag{11}$$

The N-term approximation is $S_N = \sum_{n=0}^{N} v_n$.

Two useful special cases. If $\widetilde{\mathcal{N}}[y] = \varepsilon y^2$, then

$$A_n = \varepsilon \sum_{i=0}^{n} v_i \, v_{n-i}. \tag{12}$$

If $\widetilde{\mathcal{N}}[y] = -\lambda e^{y}$, write $e^{\sum p^i v_i} = \sum p^n E_n$. Differentiating $E(p) = \exp\big(V(p)\big)$ gives $E' = V'E$, hence

$$E_0 = e^{v_0}, \qquad E_n = \frac{1}{n}\sum_{k=1}^{n} k \; v_k \, E_{n-k} \quad (n \geq 1), \qquad A_n = -\lambda E_n. \tag{13}$$

Remark 1 (Boundary conditions are automatic). Each $v_{n+1} = KA_n$ is an integral of the Green's function against a continuous function, so it satisfies (2) by property (G1). The same holds for $v_0 = F$. Consequently every partial sum S_N satisfies the boundary conditions exactly, and no constants of integration have to be determined at any order.

Remark 2 (Relation to the decomposition method). For the homotopy (8), the recursion (11) coincides with the Adomian decomposition method applied to the Green's-function form of the problem (Singh et al., 2015; Singh & Wazwaz, 2016), and the resulting algorithm is that of Wang et al. (2009): both give $v_{n+1} = KA_n$ with the same polynomials (10). The convergence result of Section 3.1 therefore applies to both methods.

2.3 The HPM–Green algorithm

Algorithm 1 (HPM–Green). *Input:* operator L, boundary conditions (2), source f, nonlinearity $\mathcal{N}$, order N.

1. Write the problem as $Ly = f + \widetilde{\mathcal{N}}[y]$ with $\widetilde{\mathcal{N}} = -\mathcal{N}$.

2. Construct $G(x, \xi)$ for L with (2), e.g. by (4).

3. Set $v_0 = F = \int_a^b G\,(\cdot, \xi) f(\xi)\,\mathrm{d}\xi$.

4. For $n = 0,1, \ldots, N-1$: compute the Adomian polynomial A_n from $v_0, \ldots, v_n$ by (10), and set $v_{n+1} = \int_a^b G\,(\cdot, \xi) A_n(\xi)\,\mathrm{d}\xi$.

5. Return $S_N = \sum_{n=0}^{N} v_n$, and estimate the error by $\|S_N - S_{N-1}\| = \|v_N\|$ or by the bounds (16) and Theorem 2.

When f and $\widetilde{\mathcal{N}}$ are polynomials and $L = \mathrm{d}^2/\mathrm{d}x^2$, all v_n are polynomials, and step 4 reduces to two antiderivatives followed by subtraction of a linear function (Lemma 1). This makes exact rational arithmetic possible.

3. Results and Discussion

3.1 Convergence analysis

We work in the complex Banach space $X = C([a, b]; \mathbb{C})$ with the norm $\|u\| = \max_{[a,b]} |u|$, and assume that G is continuous on $[a, b]^2$. This holds for the Green's functions of second-order problems; the analysis applies verbatim to any continuous kernel, so it also covers Hammerstein integral equations given directly in integral form. Let $B_R = \{u \in X : \|u\| \leq R\}$. We consider Nemytskii nonlinearities $\widetilde{\mathcal{N}}[u](x) = \varphi\big(u(x)\big)$, where φ is holomorphic on a disc $\{|z| < R'\}$ with $R' > R$ and real on the real axis. Polynomial and exponential nonlinearities are included.

Theorem 1. Suppose that

$$|\varphi(z) - \varphi(w)| \leq M|z - w| \qquad \text{for } |z|, |w| \leq R, \tag{14}$$

and let γ be as in (6) and $k = \gamma M$. Assume that there exists $\rho > 1$ such that

$$\rho k < 1 \qquad \text{and} \qquad \|F\| + \rho\, \gamma\, |\varphi(0)| \leq (1 - \rho k)\, R. \tag{15}$$

Then:

(a) Equation (5) has exactly one solution y in B_R, and y is real-valued whenever F is.

(b) The HPM terms satisfy $\|v_n\| \le R\,\rho^{-n}$ for all $n \ge 0$, and the series (9) converges uniformly on $[a,b]$ to y.

(c) The truncation error satisfies

$$\|y - S_N\| \le \frac{R\,\rho^{-N}}{\rho-1} = \frac{R\,\rho^{-(N+1)}}{1-\rho^{-1}}, \qquad N \ge 0. \tag{16}$$

Proof. *Step 1 (a contraction family).* For $p \in \mathbb{C}$ with $|p| \le \rho$, put $T_p[u] = F + pK\widetilde{\mathcal{N}}[u]$. For $u \in B_R$, by (6) and (14),

$$\|T_p[u]\| \le \|F\| + \rho\gamma(|\varphi(0)| + M\|u\|) \le \|F\| + \rho\gamma|\varphi(0)| + \rho kR \le R$$

by (15), and $\|T_p[u] - T_p[w]\| \le \rho k\|u - w\|$. Hence T_p is a contraction of the complete metric space B_R into itself, uniformly in $|p| \le \rho$. By Banach's fixed-point theorem (Kreyszig, 1978) it has a unique fixed point $v(p) \in B_R$.

Step 2 (analytic dependence on p). The Picard iterates $u_0(p) = F$, $u_{m+1}(p) = T_p[u_m(p)]$ remain in B_R and are X-valued holomorphic functions on the open disc $D_\rho = \{|p| < \rho\}$, because φ is holomorphic and K is bounded and linear. They converge to $v(p)$ uniformly on D_ρ, with rate $(\rho k)^m$. A uniform limit of vector-valued holomorphic functions is holomorphic (Rudin, 1991, Ch. 3), so v is holomorphic on D_ρ and $\|v(p)\| \le R$ there.

Step 3 (the HPM terms are Taylor coefficients). Write $v(p) = \sum_{n\ge0} p^n\,c_n$ with $c_n \in X$. The map $p \mapsto \widetilde{\mathcal{N}}[v(p)]$ is holomorphic, and by definition (10) its nth Taylor coefficient is $A_n(c_0, \dots, c_n)$. Comparing coefficients in the identity $v(p) = F + pK\widetilde{\mathcal{N}}[v(p)]$ gives $c_0 = F$ and $c_{n+1} = KA_n(c_0, \dots, c_n)$. This is exactly the recursion (11), so $c_n = v_n$ for all n.

Step 4 (estimates). By the Cauchy integral formula on the circle $|p| = r < \rho$, $\|v_n\| \le R\,r^{-n}$; letting $r \to \rho$ gives $\|v_n\| \le R\rho^{-n}$. Since $\rho > 1$, the series $\sum v_n$ converges absolutely in X. Its sum is $v(1)$, the unique fixed point of T_1 in B_R, that is, the unique solution of (5) in B_R. Summing the geometric tail gives (16). Finally, if F is real, T_1 maps the closed set of real-valued functions in B_R into itself, so its fixed point there coincides with y, and y is real. ▫

Corollary 1. If $k = \gamma M < 1$ and $\|F\| + \gamma|\varphi(0)| < (1-k)R$, then (15) holds for some $\rho > 1$ by continuity, and the conclusions of Theorem 1 hold.

Remark 3. The contraction condition $k < 1$ alone does not suffice: the operator must also map the ball B_R into itself. The estimate $\|T_1[u]\| \le \|F\| + \gamma(|\varphi(0)| + M\|u\|)$ shows that $\|F\| + \gamma|\varphi(0)| \le (1-k)R$, equivalently $(\|F\| + \gamma|\varphi(0)|)/(1-k) \le R$, is a *sufficient* condition for $T_1(B_R) \subset B_R$. It is not necessary in general, because the estimate is not sharp. Note that the whole left-hand side is divided by $1-k$; dividing only the term $\gamma|\varphi(0)|$ does not give a valid condition. The margin $\rho > 1$ in (15) controls the geometric rate of convergence of the series at $p = 1$.

Remark 4 (Scope of the analyticity assumption). The proof uses the analyticity of φ in Steps 2–4. Polynomial, exponential and trigonometric nonlinearities are covered. Nonsmooth nonlinearities such as $|y|$ or $\max(y, 0)$ are not: the Adomian polynomials (10) are not even defined for them. Nonlinearities involving derivatives of y require a different function space and are also outside the present theorem.

3.1.1 Sharper ranges by majorant series

Theorem 1 uses only a Lipschitz constant, which is a worst case over the whole ball. Much better ranges follow from the classical method of majorants. Write $\varphi(z) = \sum_{k\ge0} c_k\,z^k$ with radius of convergence $r_\varphi \in (0, \infty]$ and put $\hat{\varphi}(r) = \sum_{k\ge0} |c_k|\,r^k$ for $0 \le r < r_\varphi$.

Theorem 2 (Majorant theorem). Let $c = \|F\|$ and suppose that there is $a^* \in (0, r_\varphi)$ with

$$c + \gamma\,\hat{\varphi}(a^*) \le a^*. \tag{17}$$

Let $a(p) = \sum_{n\ge0} a_n\,p^n$ be the formal power series solution of the scalar equation $a = c + p\gamma\hat{\varphi}(a)$. Then:

(a) $\|v_n\| \le a_n$ for all $n \ge 0$;

(b) $\sum_{n\ge 0} a_n \le a^*$, so the HPM–Green series converges absolutely and uniformly on $[a,b]$; its sum y solves (5) and $\|y\| \le a^*$;

(c) $\|y - S_N\| \le \sum_{n>N} a_n \le a^* - \sum_{n\le N} a_n$.

Proof. (a) $C[a,b]$ with the supremum norm is a Banach algebra, so for any $u_0, \dots, u_n$ the nth Taylor coefficient of $\varphi(\sum p^i u_i)$ is bounded in norm by the nth coefficient of $\hat{\varphi}(\sum p^i \|u_i\|)$; this coefficient is nondecreasing in each $\|u_i\|$. We argue by induction. $\|v_0\| = c = a_0$. If $\|v_i\| \le a_i$ for $i \le n$, then $\|A_n\| \le [\hat{\varphi}(a(p))]_n$, because A_n depends only on $v_0, \dots, v_n$. Hence $\|v_{n+1}\| \le \gamma\|A_n\| \le \gamma[\hat{\varphi}(a(p))]_n = a_{n+1}$.

(b) Define power series $b_m(p)$ by $b_0(p) = c$ and $b_{m+1}(p) = c + p\gamma\hat{\varphi}(b_m(p))$. These are polynomials when φ is a polynomial, but in general (for instance for the exponential nonlinearity) they are power series. We show by induction on m that (i) b_m has nonnegative coefficients, (ii) $b_m(1) = \sum_n [b_m]_n \le a^*$, and (iii) $b_m \equiv a \pmod{p^{m+1}}$, i.e. $[b_m]_n = a_n$ for $n \le m$. For $m = 0$: $b_0 = c \ge 0$; $c \le a^*$ because $\hat{\varphi} \ge 0$ in (17); and $[b_0]_0 = c = a_0$. Assume (i)–(iii) for m. Since $\hat{\varphi}$ has nonnegative coefficients and b_m has nonnegative coefficients with $b_m(1) \le a^* < r_\varphi$, the composition $\hat{\varphi}(b_m(p))$ has nonnegative coefficients and, by rearranging a series of nonnegative terms, converges at $p = 1$ to $\hat{\varphi}(b_m(1)) \le \hat{\varphi}(a^*)$. Hence b_{m+1} satisfies (i), and $b_{m+1}(1) = c + \gamma\hat{\varphi}(b_m(1)) \le c + \gamma\hat{\varphi}(a^*) \le a^*$, which is (ii). For (iii), $b_m \equiv a \pmod{p^{m+1}}$ implies $\hat{\varphi}(b_m) \equiv \hat{\varphi}(a) \pmod{p^{m+1}}$, so $p\gamma\hat{\varphi}(b_m) \equiv p\gamma\hat{\varphi}(a) \pmod{p^{m+2}}$; since $a = c + p\gamma\hat{\varphi}(a)$, we get $b_{m+1} \equiv a \pmod{p^{m+2}}$. From (i)–(iii), $\sum_{n\le m} a_n = \sum_{n\le m}[b_m]_n \le b_m(1) \le a^*$ for every m. By (a), $\sum\|v_n\| \le a^*$, so $V(p) = \sum p^n v_n$ converges for $|p| \le 1$. The identity $V = F + pK\tilde{\mathcal{N}}[V]$ holds coefficientwise and both sides converge absolutely on $|p| \le 1$, because $\sum\|A_n\| \le \hat{\varphi}(a^*)$. Setting $p = 1$ shows that $y = V(1)$ solves (5).

(c) follows from (a) and (b). ▫

Corollary 2 (Closed-form majorants). The majorant coefficients are explicit in two important cases.

(i) If $\tilde{\mathcal{N}}[y] = \varepsilon y^2$ with $\varepsilon > 0$, then $a_n = c\,(\gamma\varepsilon c)^n C_n$, where $C_n = \binom{2n}{n}/(n+1)$ are the Catalan numbers. Condition (17) holds if and only if $4\gamma\varepsilon c \le 1$.

(ii) If $\tilde{\mathcal{N}}[y] = -\lambda e^y$ with $\lambda > 0$ and $F = 0$, then $a_n = \frac{n^{n-1}}{n!}(\gamma\lambda)^n$, the coefficients of the tree function $T(z) = -W(-z)$, where W is the Lambert function. Condition (17) holds if and only if $\gamma\lambda e \le 1$.

Proof. (i) $a = c + p\gamma\varepsilon a^2$ gives $a(p) = \left(1 - \sqrt{1 - 4p\gamma\varepsilon c}\right)/(2p\gamma\varepsilon) = c\sum_n C_n\,(p\gamma\varepsilon c)^n$, and $c + \gamma\varepsilon a^2 \le a$ is solvable if and only if its discriminant $1 - 4\gamma\varepsilon c$ is nonnegative. (ii) $a = p\gamma\lambda e^a$ is solved by $a = T(p\gamma\lambda)$, whose coefficients follow from Lagrange inversion, and $\gamma\lambda e^a \le a$ is solvable if and only if $\gamma\lambda \le \max_{a>0} ae^{-a} = 1/e$. ▫

3.1.2 Sharp convergence region: positivity and turning points

Theorems 1 and 2 give sufficient conditions. Under sign conditions that hold in many applications, the radius of convergence can be related to the solution branch itself. We use the following hypotheses, with $\kappa, s \in \{-1, 1\}$:

(S1) $\kappa\, G(x,\xi) \ge 0$ on $[a,b]^2$;

(S2) $s\, F \ge 0$ on $[a,b]$;

(S3) $\psi(z) := s\kappa\, \varphi(sz)$ has nonnegative Taylor coefficients.

Lemma 2 (Sign of the HPM terms). Under (S1)–(S3), $w_n := s\, v_n \ge 0$ on $[a,b]$ for all $n \ge 0$.

Proof. $w_0 = sF \ge 0$. Since $\varphi(\sum p^i v_i) = s\kappa\,\psi(\sum p^i w_i)$, we have $s\kappa A_n = [\psi(\sum p^i w_i)]_n$, which is a polynomial with nonnegative coefficients in $w_0, \dots, w_n$. Hence $w_{n+1} = sKA_n = (\kappa K)(s\kappa A_n) \ge 0$ by (S1) and induction. ▫

Consider the real one-parameter family

$$v = F + \mu K\widetilde{\mathcal{N}}[v], \qquad \mu \geq 0, \tag{18}$$

whose value $\mu = 1$ is the original equation. An *analytic branch* on $[0, \mu^*)$, $0 < \mu^* \leq \infty$, is a map $\mu \mapsto v(\mu) \in C[a, b]$ with $v(0) = F$ that solves (18) for $0 \leq \mu < \mu^*$ and extends to an X-valued holomorphic function on a complex neighbourhood U of $[0, \mu^*)$; we take μ^* maximal. A sufficient condition is that v be a *regular branch*, i.e. continuous with $I - \mu K\varphi'(v(\mu))$ invertible for every $\mu \in [0, \mu^*)$. Then the analytic implicit function theorem gives local holomorphic extensions, which agree on overlapping discs centred on the real axis by local uniqueness and the identity theorem.

Theorem 3 (Sharp convergence). Let φ be entire, assume (S1)–(S3), and let $v(\mu)$ be an analytic branch on $[0, \mu^*)$.

(a) If $\mu^* > 1$, the HPM–Green series converges uniformly on $[a, b]$ to $v(1)$.

(b) If $\mu^* = 1$ and $v(\mu) \to v_c$ in $C[a, b]$ as $\mu \to 1^-$, the series converges uniformly on $[a, b]$ to v_c, which solves (5).

(c) If $\mu^* < 1$ and the branch satisfies the *blow-up condition*(D) $\|\mathrm{d}v/\mathrm{d}\mu\| \to \infty$ as $\mu \to \mu^{*-}$, the series does not converge in $C[a, b]$.

In all cases the radius of convergence R of $\sum p^n v_n$ satisfies $R \geq \mu^*$, with equality when (D) holds.

Proof. *Step 0: identification.* Since I is invertible, the analytic implicit function theorem at $\mu = 0$ gives a holomorphic solution of (18) near $\mu = 0$ that is the unique solution near F; it coincides with v for small real μ. By Step 3 of the proof of Theorem 1, its Taylor coefficients at 0 are the HPM terms v_n. Hence $R > 0$ and $V(p) := \sum p^n v_n$ agrees with v near $p = 0$. Because φ is entire, V solves (18) on the whole disc $D_R = \{|p| < R\}$: both sides are holomorphic there and have the same Taylor coefficients. Let Ω be the connected component of $D_R \cap U$ that contains $[0, \min(R, \mu^*))$. By the identity theorem, $V = v$ on Ω, in particular on $[0, \min(R, \mu^*))$.

Step 1: a vector-valued Pringsheim lemma. Suppose $R < \infty$ and V has a holomorphic extension to a disc $B(R, \delta)$. We show that this is impossible. Choose $p_0 \in (0, R)$ so close to R that, for some $\eta > 0$, the disc $D(p_0, R - p_0 + \eta)$ lies in $D_R \cup B(R, \delta)$. The Taylor series of V at p_0 then converges on this disc, and its coefficients are

$$\frac{V^{(k)}(p_0)}{k!} = s \sum_{n \geq k} \binom{n}{k} p_0^{n-k} w_n,$$

where $w_n = sv_n \geq 0$ by Lemma 2. Take $r = R - p_0 + \eta/2$. For every $x \in [a, b]$ the double series $\sum_k \sum_n \binom{n}{k} p_0^{n-k} r^k w_n(x)$ has nonnegative terms and converges to $s\,V(p_0 + r)(x)$. Summing first over k gives

$$\sum_{n \geq 0} (p_0 + r)^n \, w_n(x) = s\,V(p_0 + r)(x) \leq \|V(p_0 + r)\|.$$

Hence $\|w_n\| \leq \|V(p_0 + r)\| \, (p_0 + r)^{-n}$ for all n, so $R \geq p_0 + r > R$, a contradiction. This is the classical argument of Vivanti and Pringsheim (Titchmarsh, 1939), transferred to $C[a, b]$ through the pointwise order. The only properties used are $w_n \geq 0$ and $0 \leq u \leq w \Rightarrow \|u\| \leq \|w\|$.

Step 2: $R \geq \mu^*$. If $R < \mu^*$, then $R \in [0, \mu^*) \subset U$, and v is holomorphic on a disc $B(R, \delta) \subset U$ that meets Ω. Shrinking δ so that $B(R, \delta) \cap D_R$ is connected and contained in Ω, v is a holomorphic extension of V to $B(R, \delta)$. This contradicts Step 1.

Step 3: conclusions. (a) If $\mu^* > 1$, then $R > 1$, and the series converges in $C[a, b]$ at $p = 1$ to $V(1) = v(1)$. (b) If $\mu^* = 1$, then $R \geq 1$. For $0 < p < 1$, $x \in [a, b]$ and every N, we have $\sum_{n \leq N} p^n \, w_n(x) \leq sV(p)(x) = sv(p)(x)$. Letting $p \to 1^-$ gives $\sum_{n \leq N} w_n \, (x) \leq sv_c(x)$. The partial sums at $p = 1$ are therefore nondecreasing and bounded, so they converge pointwise to some $g \leq sv_c$. Conversely, $sv(p)(x) = \sum_n p^n \, w_n(x) \leq g(x)$ for every $p < 1$, so $sv_c \leq g$. Hence $g = sv_c$, and the convergence is uniform by Dini's theorem, since g is continuous. Letting $\mu \to 1^-$ in (18) shows that v_c solves (5). (c) If the series converged in $C[a, b]$ at $p = 1$,

then $\|v_n\| \to 0$, so $R \geq 1 > \mu^*$. Then $[0, \mu^*] \subset D_R$, and by Step 0 $v = V$ on $[0, \mu^*)$. Hence $\mathrm{d}v/\mathrm{d}\mu = V'$ would stay bounded as $\mu \to \mu^{*-}$, a contradiction. The same argument shows that $R > \mu^*$ is impossible when (D) holds, so $R = \mu^*$ in that case. ▫

Remark 5 (Singular linearisation, folds and condition (D)). Condition (D) is what the proof uses. Singularity of the linearised operator $I - \mu^* K\varphi'(v_c)$ alone does not imply (D): at a transcritical or pitchfork bifurcation point, for instance, the branch may continue analytically through μ^*. Condition (D) does hold at a *nondegenerate (quadratic) fold*, i.e. when 0 is a simple eigenvalue of the linearised operator with eigenfunction ϕ, the transversality condition $\langle \phi^*, K\widetilde{N}[v_c]\rangle \neq 0$ holds for the adjoint eigenfunction ϕ^*, and the quadratic coefficient $\langle \phi^*, \mu^* K\varphi''(v_c)\phi^2\rangle$ does not vanish. In that case $v(\mu) = v_c - C\sqrt{\mu^* - \mu}\,\phi + O(\mu^* - \mu)$ with $C \neq 0$. In the examples below, (D) is verified analytically for the Bratu problem and only numerically for Examples 1 and 3.

Remark 6 (Coefficient asymptotics). Suppose that the branch ends at a nondegenerate fold in the sense of Remark 5. Suppose also that $p = \mu^*$ is the only singularity of V on the circle $|p| = \mu^*$ and that V continues analytically to a Δ-domain at μ^*. Then the transfer theorems of singularity analysis (Flajolet & Odlyzko, 1990) give

$$v_n(x) \sim c(x)\, n^{-3/2}\, (\mu^*)^{-n}, \qquad n \to \infty. \tag{19}$$

We do not prove these additional hypotheses, even for the examples; for Examples 1 and 2 the computed coefficients are consistent with (19) (Section 3.4), which supports but does not prove them.

3.2 Numerical examples

Examples 1 and 2 use $L = \mathrm{d}^2/\mathrm{d}x^2$ on [0,1] with $y(0) = y(1) = 0$, so G and $\gamma = 1/8$ are given by Lemma 1. For these two examples the terms v_n were computed in exact rational arithmetic using (11). Example 3 is a Fredholm equation with a non-separable kernel and is treated numerically. Reported errors are maxima over 201 equally spaced points in [0,1]; they are lower estimates of the supremum norm. The a priori bounds of Theorems 1 and 2 bound the supremum norm and therefore also these grid maxima.

3.2.1 Example 1: a quadratic nonlinearity

Consider

$$y'' = 1 + \varepsilon y^2, \qquad y(0) = y(1) = 0, \qquad \varepsilon > 0. \tag{20}$$

Here $f = 1$, $\widetilde{N}[y] = \varepsilon y^2$ and $\varphi(z) = \varepsilon z^2$. By Lemma 1, $F(x) = 1/2\, x(x-1)$, so $\|F\| = 1/8$. Using (12), the first terms are

$$\begin{aligned} v_0 &= -1/2\, x(1-x), \\ v_1 &= \varepsilon(x^6/120 - x^5/40 + x^4/48 - x/240), \\ v_2 &= \varepsilon^2(x^{10}/10800 - x^9/2160 + 11x^8/13440 - x^7/2016 - x^5/4800 + x^4/2880 - 11x/120960). \end{aligned}$$

Each term vanishes at $x = 0$ and $x = 1$, in accordance with Remark 1. In general $v_n = \varepsilon^n\, \hat{v}_n(x)$, where $\hat{v}_n$ is a polynomial of degree $4n + 2$.

Convergence condition. On B_R we have $|z^2 - w^2| \leq 2R|z - w|$, so $M = 2\varepsilon R$, $k = \varepsilon R/4$ and $\varphi(0) = 0$. Condition (15) becomes $1/8 \leq (1 - \rho\varepsilon R/4)R$. The choice $R = 1/4$, $\rho = 8/\varepsilon$ gives equality and $\rho k = 1/2 < 1$. Hence, for $0 < \varepsilon < 8$, problem (20) has a unique solution with $\|y\| \leq 1/4$, the HPM–Green series converges to it, and

$$\|y - S_N\| \leq \tfrac{1}{4}\, \tfrac{(\varepsilon/8)^{N+1}}{1 - \varepsilon/8}. \tag{21}$$

Numerical results. Problem (20) has no elementary closed-form solution. We therefore compare with a reference solution computed by second-order finite differences with Newton's method on meshes $h = 1/400$ and $h = 1/800$, followed by Richardson extrapolation. Repeating the computation on meshes $h = 1/800$ and $h = 1/1600$ changes the reference by at most 3.2×10^{-14} for $\varepsilon = 1$ and 1.4×10^{-13} for $\varepsilon = 4$; we use these differences as estimates of its accuracy, not as rigorous error bounds. Table (1) lists pointwise

values for $\varepsilon = 1$ on $[0, 1/2]$; the solution is symmetric about $x = 1/2$. Table (2) and Figure (1)(a) report the maximum error as a function of N for $\varepsilon = 1$ and $\varepsilon = 4$, together with the bound (21).

Table (1): Example 1, $\varepsilon = 1$: HPM–Green partial sums versus the reference solution.

x	S_1	S_2	S_4	Reference	$\lvert S_4 - y_{\mathrm{ref}}\rvert$
0.1	-0.045414825	-0.045423886	-0.045424143	-0.0454241440	2.65×10^{-10}
0.2	-0.080807466	-0.080825170	-0.080825673	-0.0808256740	5.19×10^{-10}
0.3	-0.106135925	-0.106160963	-0.106161676	-0.1061616775	7.35×10^{-10}
0.4	-0.121355200	-0.121385208	-0.121386063	-0.1213860644	8.83×10^{-10}
0.5	-0.126432291	-0.126464062	-0.126464968	-0.1264649693	9.36×10^{-10}

Table (2): Example 1: maximum error $\|S_N - y_{\mathrm{ref}}\|$ and the bound (21).

N	error, $\varepsilon = 1$	bound, $\varepsilon = 1$	error, $\varepsilon = 4$	bound, $\varepsilon = 4$
0	1.46×10^{-3}	3.57×10^{-2}	6.30×10^{-3}	2.50×10^{-1}
1	3.27×10^{-5}	4.46×10^{-3}	5.73×10^{-4}	1.25×10^{-1}
2	9.07×10^{-7}	5.58×10^{-4}	6.43×10^{-5}	6.25×10^{-2}
3	2.82×10^{-8}	6.98×10^{-5}	8.04×10^{-6}	3.13×10^{-2}
4	9.36×10^{-10}	8.72×10^{-6}	1.08×10^{-6}	1.56×10^{-2}
5	3.25×10^{-11}	1.09×10^{-6}	1.50×10^{-7}	7.81×10^{-3}
6	1.14×10^{-12}	1.36×10^{-7}	2.17×10^{-8}	3.91×10^{-3}
7	†	1.70×10^{-8}	3.21×10^{-9}	1.95×10^{-3}
8	†	2.13×10^{-9}	4.84×10^{-10}	9.77×10^{-4}
9	†	2.66×10^{-10}	7.40×10^{-11}	4.88×10^{-4}
10	†	3.33×10^{-11}	1.13×10^{-11}	2.44×10^{-4}

† At the accuracy level of the reference solution ($\approx 3 \times 10^{-14}$).

For $\varepsilon = 1$ the error decreases by a factor of roughly 30–45 per term, much faster than the guaranteed factor $8/\varepsilon = 8$; for $\varepsilon = 4$ the observed factor settles near 6.5–7, against the guaranteed factor 2. In both cases the error stays below the bound (21), as Theorem 1 predicts, but the bound is conservative.

Majorant and sharp ranges. By Corollary 2(i) with $c = \gamma = 1/8$, the majorant condition $4\gamma\varepsilon c \le 1$ gives convergence for $0 < \varepsilon \le 16$, with $\|v_n\| \le 1/8\,(\varepsilon/64)^n C_n$. The sign hypotheses hold with $\kappa = -1$ (since $G \le 0$), $s = -1$ (since $F \le 0$) and $\psi(z) = \varepsilon z^2$. Hence Theorem 3 applies with μ playing the role of ε (since $v_n = \varepsilon^n \hat{v}_n$, the series at p is the series for the parameter $p\varepsilon$ at $p = 1$), and the relevant branch is $\varepsilon \mapsto y(\cdot;\varepsilon)$ issuing from $y = F$. We computed this fold by shooting from $x = 1/2$ with fourth-order Runge–Kutta integration and maximising ε over $y(1/2)$, which gives $\varepsilon_c \approx 22.60969$ at $y(1/2) \approx -0.25496$. The analyticity of the branch on $[0, \varepsilon_c)$ and condition (D) at ε_c are supported by these computations but not proved. Under these numerically supported properties, Theorem 3 gives convergence for $0 < \varepsilon \le \varepsilon_c$ and divergence for $\varepsilon > \varepsilon_c$; the proved range is $0 < \varepsilon \le 16$.

3.2.2 Example 2: the Bratu problem

The Bratu problem

$$y'' + \lambda e^{y} = 0, \qquad y(0) = y(1) = 0, \qquad \lambda > 0, \tag{22}$$

has the exact solution (Wazwaz, 2005)

$$y(x) = -2\ln\left[\frac{\cosh\left((x-1/2)\theta/2\right)}{\cosh(\theta/4)}\right], \qquad \theta = \sqrt{2\lambda}\cosh(\theta/4). \tag{23}$$

Problem (22) has two solutions for $0 < \lambda < \lambda_c \approx 3.513830719$, one for $\lambda = \lambda_c$ and none for $\lambda > \lambda_c$ (Wazwaz, 2005). In (23) we take the smaller root θ, which gives the lower solution. Here $f = 0$, $\widetilde{N}[y] = -\lambda e^{y}$ and $F = 0$, so $v_0 = 0$. By (13), $v_n = \lambda^n \hat{v}_n(x)$ with

$$v_1 = \lambda/2\ x(1-x), \qquad v_2 = \lambda^2/24\,(x^4 - 2x^3 + x), \qquad v_3 = \lambda^3(-x^6/180 + x^5/60 - x^4/96 - x^3/144 + x/160).$$

Convergence condition. For $|z|, |w| \le R$ we have $|e^z - e^w| \le e^R|z-w|$, so $M = \lambda e^R$ and $|\varphi(0)| = \lambda$. Condition (15) reads

$$\frac{\rho\lambda}{8}(1 + Re^R) \le R, \qquad \text{i.e.} \qquad \rho\lambda \le g(R) := \frac{8R}{1+Re^R}.$$

With this choice, $\rho k = Re^R/(1+Re^R) < 1$ automatically. The function g is maximised where $R^2e^R = 1$, i.e. at $R_* \approx 0.703467$. Since $R_*e^{R_*} = 1/R_*$, the maximum value is

$$\lambda^* := g(R_*) = \frac{8R_*^2}{1+R_*} \approx 2.324043. \tag{24}$$

Consequently, for $0 < \lambda < \lambda^*$ the HPM–Green series for (22) converges to the unique solution with $\|y\| \le R_*$, which is the lower branch. The convergence rate is $\rho = \lambda^*/\lambda$, and $\|y - S_N\| \le R_*\rho^{-N}/(\rho - 1)$.

Numerical results. Table (3) compares partial sums with the exact solution (23) for $\lambda = 1$, for which $\theta = 1.5171645990507545$. Figure (1)(b) shows the maximum error for $\lambda = 1,2,3$; representative values and bounds are listed in Table (6).

Table (3): Example 2 (Bratu), $\lambda = 1$: HPM–Green partial sums versus the exact solution.

x	S_3	S_6	S_{10}	**Exact**	$\lvert S_{10} - y\rvert$
0.1	0.0497046750	0.0498453407	0.0498467863	0.0498467912	$4.91 \times 10^{-}$
0.2	0.0889160889	0.0891871264	0.0891899251	0.0891899346	$9.53 \times 10^{-}$
0.3	0.1172270750	0.1176051547	0.1176090824	0.1176090958	$1.34 \times 10^{-}$
0.4	0.1343368000	0.1347855528	0.1347902379	0.1347902539	$1.60 \times 10^{-}$
0.5	0.1400607639	0.1405342448	0.1405391974	0.1405392144	$1.70 \times 10^{-}$

Majorant and sharp ranges. By Corollary 2(ii) with $\gamma = 1/8$, the majorant condition gives convergence for $0 < \lambda \le 8/e \approx 2.943$, with $\|v_n\| \le \frac{n^{n-1}}{n!}(\lambda/8)^n$. The sign hypotheses hold with $\kappa = -1$, $s = 1$ (since $F = 0$) and $\psi(z) = \lambda e^z$, so all terms are nonnegative. The lower branch is an analytic branch that ends at a fold, which can be seen from the exact solution (23). Put $t = \theta^2$. Both y and $\Lambda(t) = t/\left(2\cosh^2\left(\sqrt{t}/4\right)\right)$ are even functions of θ, hence analytic in t, and $\Lambda'(0) = 1/2$. Moreover, $\Lambda'(t) > 0$ for $0 \le t < t_c$, where $t_c = 16\alpha^2$ and $\alpha \approx 1.199679$ is the positive root of $\alpha\tanh\alpha = 1$. Thus $t(\Lambda)$ is real-analytic on $[0, \lambda_c)$ with $\lambda_c = \Lambda(t_c) = 8(\alpha^2 - 1) \approx 3.513831$, and so is $\Lambda \mapsto y(\cdot;\Lambda)$. At t_c we have $\Lambda'(t_c) = 0 \ne \Lambda''(t_c)$, so $\mathrm{d}t/\mathrm{d}\Lambda \to \infty$ as $\Lambda \to \lambda_c^-$. Since $\partial_t y(1/2) = \partial_t\, 2\ln\cosh\left(\sqrt{t}/4\right) > 0$, also $\|\mathrm{d}y/\mathrm{d}\Lambda\| \to \infty$, while y stays bounded. Since $v_n = \lambda^n \hat{v}_n$, the family (18) at μ is the Bratu problem with parameter $\mu\lambda$, so $\mu^* = \lambda_c/\lambda$. Theorem 3 therefore gives the sharp result:

HPM–Green converges for the Bratu problem if and only if $0 < \lambda \le \lambda_c \approx 3.513831$,

and at $\lambda = \lambda_c$ it converges to the fold solution. The observed error ratios at $N = 14$, approximately 0.26, 0.52 and 0.78 for $\lambda = 1,2,3$, agree with the prediction $(\lambda/\lambda_c)(1 - 3/2N) \approx 0.25$, 0.51 and 0.76 from (19). We computed $\lambda_c = 3.5138307191$ with the same shooting procedure as in Example 1, in agreement with the known value (Wazwaz, 2005).

3.2.3 Example 3: a Fredholm equation with a non-separable kernel

To test the method on an integral equation given directly in integral form, consider the Hammerstein equation

$$u(x) = f(x) + \lambda \int_0^1 e^{-|x-t|}\, u(t)^2\, \mathrm{d}t, \qquad f(x) = 1 - \lambda(2 - e^{-x} - e^{x-1}), \qquad 0 \le x \le 1. \tag{25}$$

The function f was chosen so that $u \equiv 1$ is an exact solution for every λ. The kernel is not separable, so the method of degenerate kernels does not apply directly.

Remark 7 (Equivalent Robin problem). The function $1/2\, e^{-|x-t|}$ is the Green's function of the operator $-\mathrm{d}^2/\mathrm{d}x^2 + 1$ on [0,1] with the Robin conditions $w'(0) = w(0)$ and $w'(1) = -w(1)$. Applying this operator to (25) shows that (25) is equivalent to the Robin BVP

$$-u'' + u = 1 - 2\lambda + 2\lambda u^2, \qquad u'(0) - u(0) = -1, \qquad u'(1) + u(1) = 1.$$

Example 3 therefore also illustrates HPM–Green for an operator other than $\mathrm{d}^2/\mathrm{d}x^2$ and for Robin boundary conditions.

Convergence condition. Here $\varphi(z) = \lambda z^2$, $M = 2\lambda R$ and $\varphi(0) = 0$. The kernel norm is

$$\gamma = \max_x \int_0^1 e^{-|x-t|}\, \mathrm{d}t = 2 - 2e^{-1/2} \approx 0.78694,$$

attained at $x = 1/2$. For $0 < \lambda < 1$ we have $\|f\| = 1 - \lambda g_0$ with $g_0 = 1 - e^{-1}$. Condition (15) reads $1 - \lambda g_0 \le (1 - 2\rho\gamma\lambda R)R$. Its right-hand side is maximised at $R = 1/(4\rho\gamma\lambda)$, with maximum value $1/(8\rho\gamma\lambda)$. Hence (15) holds for some $\rho > 1$ whenever $8\gamma\lambda(1 - g_0\lambda) < 1$, that is, for

$$0 < \lambda < \lambda_3^* = \frac{8\gamma - \sqrt{64\gamma^2 - 32\gamma g_0}}{16\gamma g_0} \approx 0.17913. \tag{26}$$

For such λ, $u \equiv 1$ is the unique solution in B_R and the HPM–Green series converges to it.

Numerical results. The kernel is not polynomial, so the terms $v_{n+1} = \lambda K[\sum_{i \le n} v_i\, v_{n-i}]$ were computed numerically. Each v_n is represented by its Chebyshev interpolant of degree 40 on [0,1]. The integrals over $[0, x]$ and $[x, 1]$ are evaluated separately by 60-point Gauss–Legendre quadrature, so the kink of the kernel at $t = x$ causes no loss of accuracy. As a check, the computed $K[1]$ agrees with its exact value $2 - e^{-x} - e^{x-1}$ to 4×10^{-15}. Table (4) and Figure (1)(c) show the results for $\lambda = 0.1$, 0.2 and 0.3.

Table (4): Example 3, $\lambda = 0.1$: HPM–Green partial sums (exact solution $u \equiv 1$). The solution is symmetric about $x = 1/2$.

x	S_2	S_6	S_{10}	$\lvert S_{10} - u \rvert$
0.00	0.9984352929	0.9999969194	0.9999999906	9.45×10^{-9}
0.25	0.9981321973	0.9999963177	0.9999999887	1.13×10^{-8}
0.50	0.9980267788	0.9999961059	0.9999999881	1.19×10^{-8}

The error ratios between consecutive orders at $N = 16$ are approximately 0.25, 0.46 and 0.64 for $\lambda = 0.1,0.2,0.3$.

Majorant range. By Corollary 2(i) with $c = 1 - \lambda g_0$, convergence to $u \equiv 1$ is guaranteed for $4\gamma\lambda(1 - g_0\lambda) \le 1$, that is, for $0 < \lambda \le 0.44015$ (proved). This covers all three values $\lambda = 0.1,0.2,0.3$.

The threshold λ_t (proved). The linearisation of (25) at $u \equiv 1$ is $I - 2\lambda K$. Since $1/2\, e^{-|x-t|}$ is the Green's function of $-\mathrm{d}^2/\mathrm{d}x^2 + 1$ with the Robin conditions of Remark 7, the eigenvalues of K are $2/(1 + k^2)$, where k^2 runs through the eigenvalues of $-\mathrm{d}^2/\mathrm{d}x^2$ with these conditions. Integration by parts gives $\int_0^1 (-w'')\, w\, \mathrm{d}x = \int_0^1 w'^2\, \mathrm{d}x + w(0)^2 + w(1)^2 > 0$ for $w \neq 0$, so all k are real. The symmetric eigenfunctions $\cos\big(k(x - 1/2)\big)$ require $k\tan(k/2) = 1$, and the antisymmetric ones $\sin\big(k(x - 1/2)\big)$ require $\tan(k/2) = -k$, whose positive roots exceed π. Hence the largest eigenvalue of K is $2/(1 + k_1^2)$, where k_1 is the smallest positive root of $k\tan(k/2) = 1$, and $I - 2\lambda K$ is invertible for $0 < \lambda < \lambda_t$ and singular at

$$\lambda_t = \frac{1+k_1^2}{4} \approx 0.676763, \qquad k_1 \approx 1.306542. \tag{27}$$

Which solution is reached (numerical evidence). Hypotheses (S1)–(S3) hold with $\kappa = s = 1$ and $\psi(z) = \lambda z^2$, as long as $f > 0$, i.e. $\lambda < 1/\gamma$. By Theorem 3, the series converges to $v(1)$, where v is the analytic branch of $v = f + \mu\lambda K v^2$ issuing from f, provided this branch extends beyond $\mu = 1$. Whether $v(1)$ is the intended solution $u \equiv 1$ is not settled by the spectral argument above. We therefore tracked the branch numerically, by shooting for the equivalent Robin problem with the parameter $\mu\lambda$, at $\lambda \in \{0.1,0.3,0.5,0.6,0.65,0.7,0.8,0.9\}$. In all these cases the branch reaches beyond $\mu = 1$ before its fold (the fold values μ^* range from 3.65 at $\lambda = 0.1$ to 1.0012 at $\lambda = 0.7$). For $\lambda \le 0.65$, $u \equiv 1$ lies on the branch issuing from f, below the fold; for $\lambda \ge 0.7$ it lies on the upper branch. At $\lambda = 0.8$ the partial sums at $x = 1/2$ are 0.669721, 0.671745, 0.671906 and 0.671924 for $N = 50,100,150,200$; they approach the value $u(1/2) = 0.671926$ of the lower solution computed by shooting, not the value 1. These computations indicate that the series converges to $u \equiv 1$ for $\lambda < \lambda_t$ and to a different solution for λ somewhat above λ_t, but they cover only the tested values and do not constitute a proof. In particular, we have not proved that the fold of the branch passes through $u \equiv 1$ exactly at λ_t, nor the nondegeneracy conditions of Remark 5. The example nevertheless shows that convergence of an HPM series alone does not certify that the intended solution has been found.

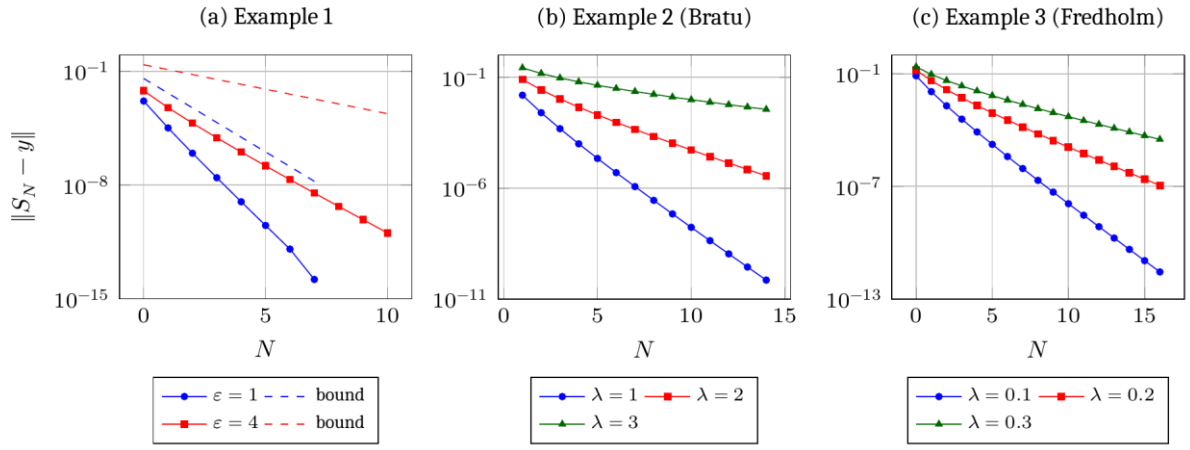


Figure (1): Maximum error of the HPM–Green partial sums S_N (semi-logarithmic scale). Dashed lines in (a) are the a priori bounds (21).

3.3 Summary of convergence ranges and error bounds

Table (5) collects the ranges given by the three theorems. Table (6) compares the corresponding a priori error bounds with the observed errors.

Table (5): Parameter ranges for convergence of HPM–Green to the intended solution. Columns 2 and 3 are proved. The last column gives the range indicated by Theorem 3 and states which ingredients are proved and which are numerical.

Problem	Thm. 1	Thm. 2	Thm. 3	Status of last column
Example 1, ε	< 8	≤ 16	$\le \varepsilon_c \approx 22.60$	fold and branch numerical
Example 2, λ	< 2.3	≤ 2.9	$\le \lambda_c \approx 3.513$	proved
Example 3, λ	< 0.1	≤ 0.4	$< \lambda_t \approx 0.676$	λ_t proved; branch numerical

Table (6): Observed maximum errors and the a priori bounds of Theorem 1 (eq. (16)) and Theorem 2 (tail $\sum_{n>N} a_n$).

Case	N	observed	Theorem 1	Theorem 2
Example 1, $\varepsilon = 4$	2	6.43×10^{-5}	6.25×10^{-2}	1.86×10^{-4}
	6	2.17×10^{-8}	3.91×10^{-3}	2.53×10^{-7}
	10	1.13×10^{-11}	2.44×10^{-4}	5.37×10^{-10}
Bratu, $\lambda = 1$	2	2.52×10^{-3}	9.84×10^{-2}	3.80×10^{-3}
	6	4.97×10^{-6}	3.37×10^{-3}	1.55×10^{-5}
	10	1.70×10^{-8}	1.16×10^{-4}	1.08×10^{-7}
	14	7.07×10^{-11}	3.96×10^{-6}	9.19×10^{-10}
Example 3, $\lambda = 0.1$	4	8.00×10^{-5}	2.18×10^{-1}	1.12×10^{-4}
	8	2.09×10^{-7}	2.11×10^{-2}	3.94×10^{-7}
	12	7.12×10^{-10}	2.05×10^{-3}	1.80×10^{-9}
	16	2.75×10^{-12}	2.00×10^{-4}	9.36×10^{-12}

The majorant bounds exceed the observed errors by factors of about 1.4 to 50, whereas the contraction bounds exceed them by factors of about 40 to 10^8.

3.4 Detecting turning points from the HPM coefficients

If (19) holds, the ratios $r_n = v_n(x_0)/v_{n-1}(x_0)$ at any point x_0 satisfy

$$\mu^* r_n = 1 - \frac{3}{2n} + O(n^{-2}).$$

Plotting r_n against $1/n$ (a Domb–Sykes plot, Domb & Sykes, 1957) therefore gives a straight line whose intercept is $1/\mu^*$. Since $v_n = \lambda^n \hat{v}_n$ in Examples 1 and 2, the terms $\hat{v}_n$ computed once yield the critical parameter directly. We use three estimators of μ^*: the plain ratio $1/r_n$; linear extrapolation in $1/n$, $1/(nr_n - (n-1)r_{n-1})$; and quadratic extrapolation through three consecutive ratios. Table (7) lists the results obtained from $\hat{v}_n(1/2)$, computed in exact rational arithmetic, and Figure (2) shows the Domb–Sykes plots.

Table (7): Turning points estimated from the first N HPM–Green coefficients, compared with the values from shooting ($\lambda_c = 3.5138307191$, $\varepsilon_c = 22.609689$).

N	Bratu: ratio	Bratu: linear	Bratu: quadratic	Ex. 1: ratio	Ex. 1: linear	Ex. 1: quadratic
10	4.1124	3.53144	3.513655	26.179	22.9232	22.64388
20	3.7942	3.51795	3.513814	24.349	22.6909	22.61393
30	3.6969	3.51563	3.513826	23.759	22.6463	22.61094
40	3.6497	3.51483	3.513829	23.468	22.6304	22.61022
60	3.6035	3.51427	3.513830	23.180	22.6190	22.60985

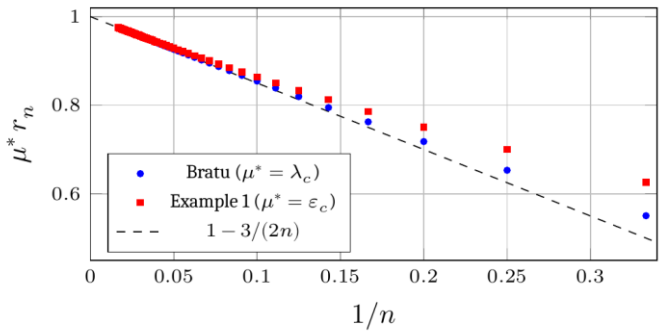


Figure (2): Domb–Sykes plot of the normalised coefficient ratios $\mu^* r_n$ at $x = 1/2$. Both sets approach the line $1 - 3/2n$ expected from (19), consistent with a square-root singularity at the fold.

With 60 terms the quadratic estimator reproduces λ_c to seven significant digits and ε_c to five. The normalised quantities $\hat{v}_n(1/2)\,(\mu^*)^n n^{3/2}$ appear to approach constants (about 0.470 for Bratu and 0.072 for Example 1 at $n = 60$), which is consistent with the exponent $-3/2$ in (19). Since (19) rests on hypotheses that we have not proved (Remark 6), the extrapolated values are numerical estimates; their agreement with the independently computed fold values supports the procedure.

3.5 Comparison with other methods

We compare HPM–Green with three alternatives. The first is *Picard–Green iteration*, $y_{m+1} = F + K\widetilde{N}[y_m]$ with $y_0 = F$. Each step applies the operator K once, exactly as each HPM term does, and in the sense of Khuri and Sayfy (2014) it is also the fixed-point form of VIM for these problems. The second is the *finite-difference method*: second-order central differences with Newton's method, without extrapolation. The third is the *Nyström method* with the trapezoidal rule and Newton's method. Table (8) reports the maximum errors for Example 1 with $\varepsilon = 4$ and for Example 3 with $\lambda = 0.1$, computed with the supplementary scripts. For Example 1 it also gives the polynomial degree of the symbolic approximations. We did not measure run times; the cost discussion below is in terms of operation counts.

Table (8): Comparison of methods for Example 1 ($\varepsilon = 4$) and Example 3 ($\lambda = 0.1$): maximum error over the 201-point grid. Nyström uses the trapezoidal rule. One HPM–Green term and one Picard–Green step each require one application of K. The orders used in each row are listed in the second column; "degree" is the polynomial degree of the exact rational approximation.

Method	Orders	1st	2nd	3rd	4th
Ex. 1: HPM–Green	$N = 2,4,6,8$	6.43×10^{-5}	1.08×10^{-6}	2.17×10^{-8}	4.84×10^{-10}
degree of S_N		10	18	26	34
Ex. 1: Picard–Green	$m = 2,3,4,5$	5.30×10^{-5}	4.91×10^{-6}	4.56×10^{-7}	4.22×10^{-8}
degree of y_m		14	30	62	126
Ex. 1: finite differences	$n = 50,100,200$	2.52×10^{-6}	6.30×10^{-7}	1.57×10^{-7}	3.94×10^{-8}
Ex. 3: HPM–Green	$N = 4,8,12$	8.00×10^{-5}	2.09×10^{-7}	7.12×10^{-10}	–
Ex. 3: Picard–Green	$m = 4,8,12$	3.61×10^{-5}	1.72×10^{-8}	8.20×10^{-12}	–
Ex. 3: Nyström	$n = 40,80,160,$	4.81×10^{-6}	1.20×10^{-6}	3.01×10^{-7}	7.52×10^{-8}

Findings.

(a) *Per application of K, Picard–Green is as accurate as HPM–Green or more accurate.* In Example 3 Picard–Green gains roughly two extra digits after twelve steps. The cost per step is not identical, however: HPM–Green order n also requires the Adomian polynomial A_n, which costs $n + 1$ products of previous terms for a quadratic nonlinearity, and n products for the exponential recursion (13), so N terms cost $O(N^2)$ products

in addition to N applications of K; a Picard–Green step costs one evaluation of $\widetilde{\mathcal{N}}$ (one product for a quadratic nonlinearity) and one application of K. Where K and the products are evaluated numerically on a fixed grid or in a fixed polynomial basis, Picard–Green is therefore the more efficient choice, and we do not claim otherwise.

(b) *In symbolic computation, HPM–Green keeps the approximations small.* The degree of S_N grows linearly ($4N + 2$ in Example 1), whereas the degree of the Picard iterates roughly doubles at each step: 126 after five steps, against 34 for S_8, which is the more accurate of the two. Since the cost of exact rational multiplication grows with the degree and with the size of the rational coefficients, this degree growth dominates the cost of symbolic Picard iteration. In this setting HPM–Green yields compact closed-form approximations.

(c) *HPM–Green separates the parameter.* Since $v_n = \varepsilon^n \hat{v}_n(x)$ in Example 1 and $v_n = \lambda^n \hat{v}_n(x)$ in Example 2, one computation of $\hat{v}_0, \dots, \hat{v}_N$ gives an explicit approximation $S_N(x;\varepsilon) = \sum_n \varepsilon^n \hat{v}_n(x)$ that is valid for the whole guaranteed parameter range at once. Picard iterates do not have this structure.

(d) *Grid-based methods* (finite differences, Nyström) are general and robust, but in their basic form they converge only algebraically, as $O(h^2)$, and return discrete values. In these smooth problems, HPM–Green reaches a given accuracy with few terms and returns an analytic expression together with an a priori error bound.

(e) *HPM in differential form* produces the same terms for Examples 1 and 2, because solving $v_{n+1}'' = A_n$ with $v_{n+1}(0) = v_{n+1}(1) = 0$ is exactly the application of K. The Green's function performs the boundary fitting once and for all. For the Robin problem of Remark 7, fitting two constants at every order by hand is noticeably more laborious.

3.6 Discussion

Advantages. Because the kernel carries the boundary data, (i) every approximation satisfies the boundary conditions exactly, (ii) each order requires a single integration, (iii) the integral form allows convergence to be characterised precisely, from sufficient conditions (Theorems 1 and 2) to a characterisation of the radius of convergence in terms of the solution branch (Theorem 3), (iv) the terms can be used to estimate the turning point of the solution branch, and (v) in symbolic computation the approximations stay compact and the parameter separates.

Limitations. (i) The method requires a Green's function in closed form, or an accurate way to evaluate it. (ii) Theorem 3 needs the sign conditions (S1)–(S3); without them, singularities off the positive real axis may limit convergence, and only Theorems 1 and 2 apply. (iii) The plain series follows only the analytic branch issuing from F; upper branches, such as that of the Bratu problem, are not reached, and, as Example 3 shows, the branch followed need not contain the intended solution. (iv) For Examples 1 and 3, the analyticity of the branches, condition (D) at the folds, the fold values and, in Example 3, the position of $u \equiv 1$ on the branch were established numerically, not proved; only the Bratu results and the spectral value λ_t are proved. (v) The coefficient asymptotics (19) underlying the Domb–Sykes estimates rest on unproved hypotheses. (vi) In purely numerical computations, Picard–Green iteration converges at least as fast per application of K.

Further work. Padé approximants in p or λ could provide an analytic continuation of the series beyond its radius of convergence. Since a fold is a square-root branch point, the continuation is two-valued near it, and which branch a Padé approximant represents depends on the continuation path and on how the approximant places its poles and zeros along a cut. Reaching the upper branch of the Bratu problem in this way is therefore not automatic and would require a separate analysis, for example in the variable $\sqrt{\lambda_c - \lambda}$. Other extensions are homotopies with convergence-control parameters, higher-order and non-self-adjoint operators, and systems of BVPs.

4. Conclusion

We have studied when, and to what, the Green's function–homotopy perturbation series converges for nonlinear BVPs and Hammerstein integral equations. A contraction theorem gives existence, uniqueness

and geometric error bounds. A majorant theorem reduces convergence to a scalar inequality and yields closed-form bounds of Catalan and Lambert-W type, which in our examples exceed the observed errors by less than two orders of magnitude. Under sign conditions, a Pringsheim argument shows that the radius of convergence in the embedding parameter is at least the extent of the analytic solution branch, and equals it at a fold; for the Bratu problem this proves that HPM–Green converges if and only if $\lambda \leq \lambda_c = 8(\alpha^2 - 1)$. For Example 1 and Example 3 the corresponding ranges rest on numerical branch tracking. In Example 3 we identified in closed form the value λ_t at which the linearisation at the intended solution becomes singular, and found numerically that beyond it the series converges to a different solution. Finally, the computed coefficients are consistent with the asymptotics expected at a quadratic fold, which allows the turning point to be estimated from the series itself. These results delimit the domain in which HPM–Green can be used with confidence and indicate where numerical verification remains necessary.

Declarations

Competing interests. The author declares that she has no competing interests.

Funding. This research received no external funding.

Data availability. No external data were used. All numerical results were generated by the computations described in the paper; the Python scripts (with instructions and library versions) that regenerate all numerical values in Tables (1)–(8) and the data plotted in Figures (1)–(2) are provided as supplementary material.

Use of generative AI. During the preparation of this work the author used Claude (Anthropic) to assist with drafting and editing the text, carrying out symbolic and numerical computations, and checking bibliographic data. The author reviewed and verified all content, including proofs, computations and references, and takes full responsibility for the published article.

References

Abbaoui, K., & Cherruault, Y. (1994). Convergence of Adomian's method applied to differential equations. *Computers & Mathematics with Applications*, *28*(5), 103–109. https://doi.org/10.1016/0898-1221(94)00144-8

Abbasbandy, S. (2006). Numerical solutions of the integral equations: Homotopy perturbation method and Adomian's decomposition method. *Applied Mathematics and Computation*, *173*(1), 493–500. https://doi.org/10.1016/j.amc.2005.04.077

Abdelrazec, A., & Pelinovsky, D. (2011). Convergence of the Adomian decomposition method for initial-value problems. *Numerical Methods for Partial Differential Equations*, *27*(4), 749–766. https://doi.org/10.1002/num.20549

Adomian, G. (1994). *Solving frontier problems of physics: The decomposition method*. Kluwer Academic Publishers. https://doi.org/10.1007/978-94-015-8289-6

Al-Hayani, W., & Casasús, L. (2005). The Adomian decomposition method in turning point problems. *Journal of Computational and Applied Mathematics*, *177*(1), 187–203. https://doi.org/10.1016/j.cam.2004.09.016

Ayati, Z., & Biazar, J. (2015). On the convergence of homotopy perturbation method. *Journal of the Egyptian Mathematical Society*, *23*(2), 424–428. https://doi.org/10.1016/j.joems.2014.06.015

Badredine, T., Abbaoui, K., & Cherruault, Y. (1999). Convergence of Adomian's method applied to integral equations. *Kybernetes*, *28*(5), 557–564. https://doi.org/10.1108/03684929910277779

Biazar, J., & Ghazvini, H. (2009). Convergence of the homotopy perturbation method for partial differential equations. *Nonlinear Analysis: Real World Applications*, *10*(5), 2633–2640. https://doi.org/10.1016/j.nonrwa.2008.07.002

Cherruault, Y. (1989). Convergence of Adomian's method. *Kybernetes*, *18*(2), 31–38. https://doi.org/10.1108/eb005812

Domb, C., & Sykes, M. F. (1957). On the susceptibility of a ferromagnetic above the Curie point. *Proceedings of the Royal Society of London. Series A*, *240*(1221), 214–228. https://doi.org/10.1098/rspa.1957.0078

Flajolet, P., & Odlyzko, A. (1990). Singularity analysis of generating functions. *SIAM Journal on Discrete Mathematics*, *3*(2), 216–240. https://doi.org/10.1137/0403019

Ghasemi, M., Tavassoli Kajani, M., & Babolian, E. (2007). Numerical solutions of the nonlinear Volterra–Fredholm integral equations by using homotopy perturbation method. *Applied Mathematics and Computation*, *188*(1), 446–449. https://doi.org/10.1016/j.amc.2006.10.015

He, J.-H. (1999). Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*, *178*(3–4), 257–262. https://doi.org/10.1016/S0045-7825(99)00018-3

He, J.-H. (2000). A coupling method of a homotopy technique and a perturbation technique for non-linear problems. *International Journal of Non-Linear Mechanics*, *35*(1), 37–43. https://doi.org/10.1016/S0020-7462(98)00085-7

Jyoti, & Singh, M. (2022). An iterative technique based on HPM for a class of one dimensional Bratu's type problem. *Mathematics and Computers in Simulation*, *200*, 50–64. https://doi.org/10.1016/j.matcom.2022.04.007

Kafri, H. Q., & Khuri, S. A. (2016). Bratu's problem: A novel approach using fixed-point iterations and Green's functions. *Computer Physics Communications*, *198*, 97–104. https://doi.org/10.1016/j.cpc.2015.09.006

Khuri, S. A., & Louhichi, I. (2018). A novel Ishikawa–Green's fixed point scheme for the solution of BVPs. *Applied Mathematics Letters*, *82*, 50–57. https://doi.org/10.1016/j.aml.2018.02.016

Khuri, S. A., & Sayfy, A. (2014). Variational iteration method: Green's functions and fixed point iterations perspective. *Applied Mathematics Letters*, *32*, 28–34. https://doi.org/10.1016/j.aml.2014.01.006

Kreyszig, E. (1978). *Introductory functional analysis with applications*. Wiley.

Rach, R. (1984). A convenient computational form for the Adomian polynomials. *Journal of Mathematical Analysis and Applications*, *102*(2), 415–419. https://doi.org/10.1016/0022-247X(84)90181-1

Rudin, W. (1991). *Functional analysis* (2nd ed.). McGraw-Hill.

Sawangtong, W., & Sawangtong, P. (2019). Green's function homotopy perturbation method for the initial-boundary value problems. *Advances in Difference Equations*, *2019*, Article 419. https://doi.org/10.1186/s13662-019-2350-7

Singh, R., Nelakanti, G., & Kumar, J. (2015). Approximate solution of two-point boundary value problems using Adomian decomposition method with Green's function. *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, *85*(1), 51–61. https://doi.org/10.1007/s40010-014-0170-4

Singh, R., & Wazwaz, A.-M. (2016). Numerical solutions of fourth-order Volterra integro-differential equations by the Green's function and decomposition method. *Mathematical Sciences*, *10*(4), 159–166. https://doi.org/10.1007/s40096-016-0190-0

Stakgold, I., & Holst, M. (2011). *Green's functions and boundary value problems* (3rd ed.). Wiley. https://doi.org/10.1002/9780470906538

Titchmarsh, E. C. (1939). *The theory of functions* (2nd ed.). Oxford University Press.

Tomar, S., Dhama, S., Ramos, H., & Singh, M. (2023). An efficient technique based on Green's function for solving two-point boundary value problems and its convergence analysis. *Mathematics and Computers in Simulation*, *210*, 408–423. https://doi.org/10.1016/j.matcom.2023.03.015

Wang, Y.-G., Song, H.-F., & Li, D. (2009). Solving two-point boundary value problems using combined homotopy perturbation method and Green's function method. *Applied Mathematics and Computation*, *212*(2), 366–376. https://doi.org/10.1016/j.amc.2009.02.036

Wazwaz, A.-M. (2005). Adomian decomposition method for a reliable treatment of the Bratu-type equations. *Applied Mathematics and Computation*, *166*(3), 652–663. https://doi.org/10.1016/j.amc.2004.06.059