CONTINUED FRACTION AND BINARY TREE GRAPHS

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Abstract

A continued fraction is a way of representing a real number by a sequence of integers. In this paper, we display an explanation from the continued fraction expansion in a more general state, and we present a new method to think about these continued fractions using tree graphs. Continued fractions, binary tree graphs, the topological index Z, and the Euclidean division algorithm are combined. In fact, we found a new combinatorial realization of the continued fractions with the binary trees and number of connected components of binary trees. Our aim is to show how this realization reflects the convergence of the continued fractions, the topological index Z, and as well as the Euclidean division algorithm. We think that this different perspective can be useful because the continued fraction depends on the order of vertices, which are the set of all positive rational numbers. Thus, the choice of the right sequence for vertices of binary tree has a significant impact on the build of continued fraction. The connection between binary tree, sub binary trees, and continued fractions will be explored. Findings are to establish results on sums of vertices, palindromic continued fractions.

Keywords: Continued Fraction; Binary Tree Graph Method; Topological Index; Euclidean Division Algorithm.

Introduction

In graph theory, a tree $T(G) = (V, E)$ is an undirected graph in which any two vertices (V) are connected by exactly one edge (E). Also, it is a connected graph that has no cycles and often has a pyramid shape, or a hierarchical data structure composed of vertices. The most used tree in computing is the binary tree (Valiente, 2002). The binary tree is a nonlinear data structure that consists of vertices that have at most two children, and each child of a vertex is designated as its left or right child. It is a finite set of elements that is either empty or contains a root vertex and left- and right-subtrees that are also binary trees. This means, each vertex contains a "left" pointer, and a "right" pointer, where the root refers to the topmost vertex in the tree, and the left and right pointers recursively refer to smaller "subtrees" on both sides. A null pointer represents a binary tree with no elements - the empty tree (Parlante, 2001).

The continued fractions theory has gained great importance during the past two hundred years. The key concept of this theory is on real numbers and is to give an approximation of different real numbers with rational numbers. A continued fraction is an expression gained via an iterative process of representing a number as the sum of its integer part and the reciprocal of another number, then this other number is written as the sum of its integer part and another reciprocal, and so on (Mennou, et al, 2021).

In our work, we studied binary tree graphs from an abstract point of view and established binary tree graphs and connected the graphs which reflect the relations between the continued fraction, topological index z of binary tree, and the Euclidean division algorithm. We established a bijection between the continued fractions $[a_1, a_2, a_3, \dots a_n]$ and the binary tree graphs $\mathrm{T}(G)$ $[v_1, v_2, v_3, \dots v_n$, such that each continued fraction corresponds to a binary tree and the connected components of the binary tree, and vice versa. We explained what this combinatorial realization of the continued fractions means. We provided the basic concept of continued fractions in Section 2. Section 3 introduces a combinatorial interpretation of the binary tree, the topological index Z, and its associated binary tree continued fraction. In Section 4, we introduced the combination of the Euclidean division algorithm and its relation to the binary tree on rational numbers. In Section 5, we provided the main application and results discussions (i.e. the proof the theorem to confer clarification of why this combinatorial interpretation of continued fractions is so interesting). The conclusion directions are outlined in Section 6.

Continued Fraction

Definition 1: An expression of the form

$$\boldsymbol{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + ..}}}} \qquad (\mathbf{1})$$

is called a continued fraction. In general, all the numbers $a_1, a_2, a_3, \dots b_1, b_2, b_3, \dots.$ may be any real or complex numbers, and the number of terms may be finite or infinite. In this paper, we will confine our discussion to simple continued fractions (H. S. Wall, 1948; C. D. Olds Sun Jose State College), which have the following formula

$$\boldsymbol{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}} \qquad (\mathbf{2})$$

A much more convenient manner of writing (2) is:

$$\boldsymbol{a_1 + \frac{1}{a_{2} + } \frac{1}{a_3 +} \cdots \frac{1}{+a_n}} \qquad (\mathbf{3})$$

Let us start with an easy example.

Example 1. If we are given a rational number $\frac{30}{13}$, where $\frac{30}{13} = 2 + \frac{4}{13}$

The integer 2 is called the integer part of $\frac{30}{13}$ and the rational number $\frac{4}{13}$

is the remainder. Another way of writing the above equation would be

$$\mathbf{\frac{30}{13} = 2 + \frac{1}{(\frac{13}{4})}}$$

Although it may seem strange at first sight, I prefer this way for that now we can continue our procedure with the fraction $\frac{13}{4}$. This number is bigger than 3 and smaller than 4. In fact, we have

$$\mathbf{\frac{13}{4} = 3 + \frac{1}{4}}$$

which means that

$$\mathbf{\frac{30}{13} = 2 + \frac{1}{3 + \frac{1}{4}}} \qquad (\mathbf{4})$$

Note that now the numerator of the remainder $\frac{1}{4}$ is equal to 1. Therefore, if we repeat the same procedure again, we will replace a fraction $\frac{1}{4}$ by 1 divided by its inverse $\frac{4}{1}$, but this would not change anything, since $\frac{4}{1} = 4$, obviously. Thus, we can stop our construction as soon as the numerator of the remainder is 1. The expression in (1) is called the continued fraction expansion of $\frac{30}{13}$. Since all numerators are equal to 1, we will usually just write $[2, 3, 4]$ for the right-hand side of (1). There is nothing special here about the integers 30 and 13. We can compute such a continued fraction expansion for any rational number $\frac{p}{1q}$, although we might need more than just two steps.

The Binary Tree Graphs of Continued Fraction

A common execution of binary trees uses vertices, which start with one vertex named a root, and add another vertex to the right or to the left, or on both sides. Then another is to the right or the left of the previous one and so on. A vertex that has no left and right vertex is called Leaf or a vertex with empty left and right subtrees. We are going to build binary trees out of vertices and edges. Take a certain number of vertices $v_i,\ i = 1,2, \dots, n$ and start by laying up a first vertex (a root). Then place down a second vertex either to the left or to the right of the first vertex and connect them by edge (path). You have the choice here; you can place down two vertices on both sides (left and right). Then place the third vertex either to the left or to the right of the second vertex. Again, you must choose. Continue by this method until you have used all the vertices and edges (Valiente, 2002; Parlante, 2001). We will define a "root-to-leaf path" in order to be a sequence of vertices in a tree starting with the root vertex and proceeding downward to a leaf (a vertex with no children). We will say that an empty tree contains no root-to-leaf paths (Parlante, 2001). The following binary tree in Figure (1) has exactly five root-to-leaf paths.

Figure (1): Example of a Binary Tree.

Now: Let us give a combinatorial interpretation of continued fractions. Every positive rational number q may be expressed as a continued fraction of the form

$$\boldsymbol{q = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}} = [a_0; a_1, a_2, .., a_n]} \qquad \textbf{(5)}$$

Where n and a_0 are non-negative integers, and each subsequent coefficient a_i is a positive integer. This representation is not unique because one has

$$\boldsymbol{[a_0; a_1, a_2, .., a_{n-1}, 1] = [a_0; a_1, a_2, .., a_{n-1} + 1]}$$

The $n - th$ convergent of the continued fraction expansion of q is given by

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}} = [a_0; a_1, a_2, .., a_n]$$

It is well known that the p_n and q_n satisfy the recurrence relation:

$$p_{-1} = 1 \,, p_{-2} = 0 \,,\; p_n = a_n p_{n-1} + p_{n-2} \,, \text{for } n \geq 0.$$

$$q_{-1} = 0 \,, q_{-2} = 1,\; q_n = a_n q_{n-1} + q_{n-2} \,, \quad \text{for } n \geq 0.$$

The binary tree contains every positive rational number exactly once, so does this sequence, where the denominator of each fraction equals the numerator of the next fraction in the sequence. Thus, each positive rational number q occurs as a vertex and has one outgoing edge to another vertex, its parent. Recall that a binary tree T is a connected graph consisting of a finite sequence of vertices $[v_1, v_2, v_3, \ldots v_n]$ with n ≥ 1, such that v_i andv_j share exactly one edge e_{ij} and this edge is either the left edge of v_i or the right edge of v_j, for each $i = 1, \ldots, n$.

Notice that, for every rooted tree one can associate a continued fraction in a natural way, as exemplified in Figure (2). We call this a binary tree continued fraction (Spier, 2020).

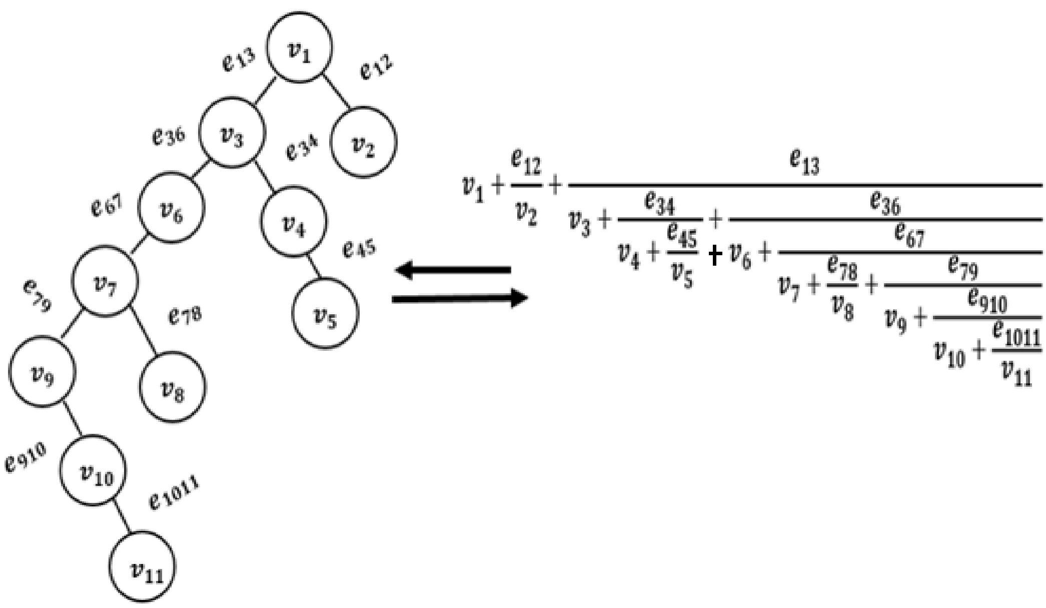


Figure (2): A Rooted Binary Tree and its Associated Tree Continued Fraction.

Every real number can be expressed as a generalized of continued fraction expansion of form q given by:

$$\boldsymbol{q = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{\ddots + \cfrac{b_n}{a_n}}}}}}$$

We suppose that all numbers $a_0, a_1, a_2, \dots a_n$ and $b_1, b_2, b_3, \dots b_n$ are positive integers.

So, the $n-th$ convergent of ${}^{p_n}/_{q_n}$ is given by:

$$\boldsymbol{\frac{p_n}{q_n} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{\ddots + \cfrac{b_n}{a_n}}}}}}$$

At this point, p_n and q_n satisfy the recurrence relation:

$$\boldsymbol{p_1 = a_1\ ,\ p_2 = a_2 p_1 + 1\ ,\ p_n = a_n p_{n-1} + p_{n-2}\ ,\text{for } n \geq 2.}$$

$$\boldsymbol{q_1 = 1\ ,\ q_2 = a_2\ ,\ q_n = a_n q_{n-1} + q_{n-2}\ ,\text{ for } n \geq 2.}$$

Now, if we go back to the example of a binary tree graph in Figure (2), the number of subsequence of vertices and edges is equal to the numerators and denominators of the continued fraction, where each integer a_i corresponds to the vertices and b_i to the edges of binary tree,

$$\mathbf{T}(\boldsymbol{G})[\boldsymbol{v_1, v_2, v_3, \dots v_n; e_1, e_2, \dots, e_{n-1}}] = [\boldsymbol{a_0, a_1, a_2, \dots a_n; b_1, b_2, b_3, \dots b_n}]$$

In graph theory, a connected component of an undirected graph is a subgraph in which each pair of vertices relates to each other via an edge (path). The subgraphs $T_1, \dots, T_n$, are connected components of the binary tree, which can be obtained by removing edges of the tree, for $i = 1, \dots, n - 1$. The connected components binary tree from the binary tree in Figure (1) above are:

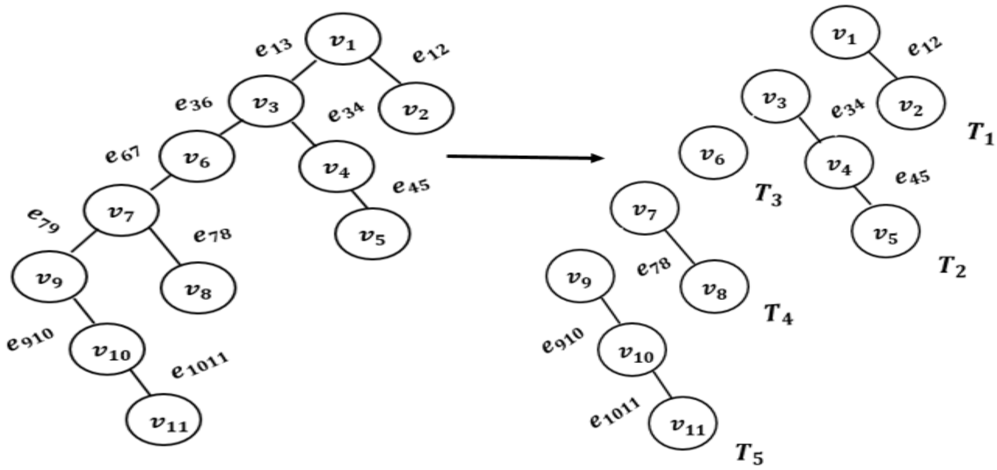


Figure (3): Binary Tree and Connected Components of Binary Tree of Continued Fraction.

Lemma: 1. If $\mathrm{T}_1, \ldots, \mathrm{T}_n$, are connected components of T(G), where $T_1(G) \cap T_2(G) \ldots \cap T_n(G) = \emptyset$ then

$$\mathbf{T}(\mathbf{G}) = \bigcup_{\mathbf{i}=1}^{\mathbf{n}} \mathbf{T}_{\mathbf{i}}(\mathbf{G}), \quad \boldsymbol{i} = \mathbf{1}, \ldots . \boldsymbol{n}$$

In Figure (4) we have Five connected components $T_1(G), \ldots, T_5(G)$, where $T_1(G) = (v_1, v_2)$, $T_2(G) = (v_3, v_4, v_5)$, $T_3(G) = (v_6)$, $T_4(G) = (v_7, v_8)$, $T_5(G) = (v_9, v_{10}, v_{11})$, and $T_1(G) \cap T_2(G) \cap T_3(G) \cap T_4(G) \cap T_5(G) = \emptyset$.

Consequently, by lemma 1

$$\boldsymbol{T}(\boldsymbol{G}) = \bigcup_{\boldsymbol{i}=1}^{\mathbf{5}} \boldsymbol{T}_{\boldsymbol{i}}(\boldsymbol{G})$$

$$\boldsymbol{T}(\boldsymbol{G}) = \boldsymbol{T}_{\mathbf{1}}(\boldsymbol{G}) \cup \boldsymbol{T}_{\mathbf{2}}(\boldsymbol{G}) \cup \boldsymbol{T}_{\mathbf{3}}(\boldsymbol{G}) \cup \boldsymbol{T}_{\mathbf{4}}(\boldsymbol{G}) \cup \boldsymbol{T}_{\mathbf{5}}(\boldsymbol{G})$$

$$[\boldsymbol{v}_{\mathbf{1}}, \boldsymbol{v}_{\mathbf{2}}, \boldsymbol{v}_{\mathbf{3}}, \ldots \boldsymbol{v}_{\mathbf{11}}] = [\boldsymbol{v}_{\mathbf{1}}, \boldsymbol{v}_{\mathbf{2}} \cup \boldsymbol{v}_{\mathbf{3}}, \boldsymbol{v}_{\mathbf{4}}, \boldsymbol{v}_{\mathbf{5}} \cup \boldsymbol{v}_{\mathbf{6}} \cup \boldsymbol{v}_{7}, \boldsymbol{v}_{\mathbf{8}} \cup \boldsymbol{v}_{\mathbf{9}}, \boldsymbol{v}_{\mathbf{10}}, \boldsymbol{v}_{\mathbf{11}}].$$

2. If $e_{ij} = v_i v_j$ is an edge of a graph T(G), then $\mathrm{T(G)} = \mathrm{T(G)} - e_{ij} + \mathrm{T(G)} - \{v_{\mathrm{i}}, v_j\}$.

3. If v_i is a vertex of a graph $T(G)$, then $T(G) = T(G) - v_i + \sum_{v_i v_j} T(G) - \{v_i v_j\}$, where the summation extends over all vertices adjacent to v_i.

The binary tree $\mathrm{T}(G)[v_1, v_2, v_3, \ldots v_n; e_1, e_2, \ldots, e_{n-1}]$ of the continued fraction $[a_0, a_1, a_2, \ldots a_n; b_1, b_2, b_3, \ldots b_n]$ is:

$$(\underbrace{v_1 v_2}_{a_1}, \underbrace{v_3, v_4, v_5}_{a_2}, \underbrace{v_6}_{a_3}, \underbrace{v_7, v_8}_{a_4}, \underbrace{v_9, v_{10}, v_{11}}_{a_5}; (\underbrace{e_{13}}_{b_1}, \underbrace{e_{36}}_{b_2}, \underbrace{e_{67}}_{b_3}, \dots, \underbrace{e_{79}}_{b_4})$$

Note: a binary tree that denoted by $\mathrm{T}(G)[v_1, v_2, v_3, \dots v_n]$, is a tree in which all the vertices are within distance 1 of a central path. In addition, a tree containing a path graph such that every edge has one or more endpoints in that path. So, in example of binary tree in Figure (3) If $v_1 = \cdot \cdot \cdot = v_n = 1$, $T(G) = (1, \dots, 1)$ is a path graph. Komatsu in (2020) have demonstrated that for $n \geq 1$ $Z(T_i(G)(v_1, v_2, v_3, \dots v_{n-1}) = p_{n-1}$, the integer Z(G) is the topological index that defined as:

$$Z = \sum_{R=o}^{m} p(G, R)$$

Where p (G, R) is the number of methods for selecting R disjoin edges from G, for more information about topological index Z see (Komatsu, 2020; Hosoya, 2007).

Theorem 1: for n ≥ 1, $Z(T_i(G)(v_1, v_2, v_3, \dots v_{n-1}) = p_{n-1}$

where $p_{n-1}$is the numerator of the convergent of the continued fraction expansion

$$\frac{p_{n-1}}{q_{n-1}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1}}}}}}$$

with $gcd(p_{n-1}, q_{n-1}) = 1$, $a_i \geq 1$ $(0 \leq i \leq n-1)$.

Corollary 1: Special cases with recurrence relations, if $v_1 = \cdot \cdot \cdot = v_n = a$, and $e_1 = \cdot \cdot \cdot = e_{n-1} = b$, where a and b are positive integers. Then, for a positive integer n consider the following sequence:

$$Z((T_i(G)\ \underbrace{(v_1, v_2, \dots \dots, v_n}_{n}; \underbrace{e_1, e_2, \dots \dots, e_{n-1}}_{n-1})$$

$$= Z((T_i(G)\ \underbrace{(\mathbf{a}, \mathbf{a}, \dots \dots, \mathbf{a}}_{n}; \underbrace{\mathbf{b}, \mathbf{b}, \dots \dots, \mathbf{b}}_{n-1})$$

$$= u_{n+1} = au_n + bu_{n-1}, with\ u_0 = 0, u_1 = 1$$

Example 2: the example of a binary tree in Figure (3) above $\mathrm{T}(G)[v_1, v_2, v_3, \dots v_n; e_1, e_2, \dots, e_{n-1}]$ the topological index Z(G) is Z $(\mathrm{T}(G)[2,3,1,2,3; 1, 1,11])$. Consequently, the corresponding continued fractions are:

$$2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}}} = [2,3,1,2,3] = \frac{84}{37}$$

Then, the topological index Z is given by Z (T(G)[2,3,1,2,3; 1, 1,11]) = 84.

We established a bijection between the topological index Z(TG), continued fractions $[a_1, a_2, a_3, ., a_n]$, and binary tree $\mathrm{T}(G)[v_1, v_2, v_3, \dots v_n; e_1, e_2, \dots., e_{n-1}]$, where the connected components of binary tree in Figure 3 with v vertices determined by the subsequence of binary trees, such that the number of connected components of the binary tree equals the numerator of the continued fraction

$$[\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \boldsymbol{a_4}, \boldsymbol{a_5}] = [\boldsymbol{T_1}, \boldsymbol{T_2}, \boldsymbol{T_3}, \boldsymbol{T_4}, \boldsymbol{T_5}] = [2,3,1,2,3].$$

Example 3: For more understanding, let us go back to the example 1 of continued fraction $[2,3,4] = \frac{30}{13}$, we can establish the binary tree, the connected components of binary tree, and topological index Z(T(G) from this continued fraction, where

$$2+\cfrac{1}{3+\cfrac{1}{4}} = [2,3,4] = \frac{30}{13}$$

Hence, the topological index Z $(\mathrm{T}(G)[v_1, v_2, v_3, \dots v_n; e_1, e_2, \dots., e_{n-1}])$ = Z $(\mathrm{T}(G)[2,3,4; 1, 1]) = 30$, So, $[a_1, a_2, a_3]$ =[2,3,4] $= [T_1, T_2, T_3]$, where $T_1 = (v_1, v_2)$, $T_2 = (v_3, v_4, v_5)$, and $T_3 = (v_6, v_7, v_8, v_9)$. $\mathrm{T}(G) = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)$. Then, as a result the binary tree and connected components of the binary tree can be established as shown in Figure (4).

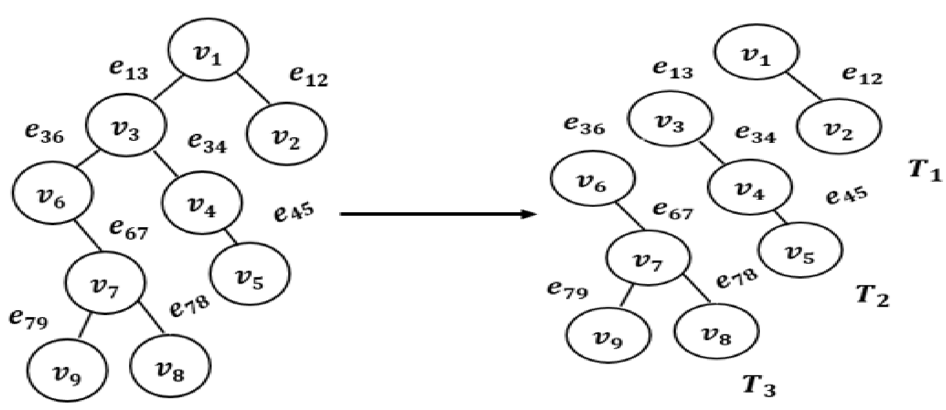


Figure (4): Binary Tree and Connected Components of Binary Tree.

Let us prove the following theorem in order to confer clarification of why this combinatorial interpretation of continued fractions is so interesting.

Theorem 2. If $T_i(G)$ denotes the number of connected components of binary tree of $T(G)$ then

$$[\boldsymbol{a_1, a_2, a_3, \dots a_n}] = \frac{\boldsymbol{T_i(G[v_1, v_2, v_3, \dots v_n])}}{\boldsymbol{T(G[v_1, v_2, v_3, \dots v_k])}}$$

and the right-hand side is a reduced fraction.

Convergence. Then $n-th$ convergent of the continued fraction $[a_1, a_2, a_3, \dots a_n]$ is the continued fraction $[a_1, a_2, a_3, \dots a_n]$ for $1 \leq \ n \leq \ k$. By Theorem 2, we have that the numerator of $n-th$ convergent is the number of connected components of tree $T_i(G)[v_1, v_2, \dots v_n]$ of $T(G)[v_1, v_2, \dots, v_k]$ and the denominator of the $n-th$ convergent is the number of the vertices of $T(G)[a_1, a_2, a_3, \dots a_n]$. It is well known that the $n-th$ convergent of the continued fraction $T_i[a_1, a_2, a_3, \dots a_k]$ is equal to $^{p_n}/_{q_n}$.

Theorem 3. For every convergent $^{p_n}/_{q_n}$ of a rational number $^{q}/_{b}$:

$$\frac{\boldsymbol{p_n}}{\boldsymbol{q_n}} = \frac{\boldsymbol{a_n p_{n-1} + p_{n-2}}}{\boldsymbol{a_n q_{n-1} + q_{n-2}}} \qquad (\mathbf{6})$$

For $0 \leq n \leq k$, defining $p_{n-1} = 1$; $q_{n-1} = 0$.

Bracha in (1974) have shown that for a limited class of functions, such as quadratic or cubic equations, a solution can be approximated by a continued fraction of the form

$$\frac{\boldsymbol{A_i}}{\boldsymbol{B_i}} = \frac{\boldsymbol{q_1}}{\boldsymbol{b_1 +}}\frac{\boldsymbol{q_2}}{\boldsymbol{b_2 +}} \dots . \frac{\boldsymbol{q_n}}{\boldsymbol{b_n + f_{n+1}}} \qquad (\mathbf{7})$$

where A_n and B_n are determined from the recursion

$$\boldsymbol{A_i = b_i A_{n-1} + q_i A_{n-2}}$$

$$\boldsymbol{B_i = b_i B_{n-1} + q_i B_{n-2} \quad i = 2, 3, \dots \dots} \quad (\mathbf{8})$$

with initial values:

$$\boldsymbol{A_0 = 0}, \quad \boldsymbol{A_1 = q_1}$$

$$\boldsymbol{B_0 = 1}, \quad \boldsymbol{B_1 = b_1}.$$

Following the analysis of Wynn (P. Wynn, 1964; G. H. Hardy and E. M. Wright, 1960), we define a continued fraction as a sequence of bilinear transformations of the form:

$$u_n = \frac{\alpha_{n+1}}{1 + u_{n+1}} \qquad n = 1, 2, \dots, n-1, \qquad (9)$$

where $f_n(x)$ is a function of x and p_k, q_k are constants, the resulting continued fraction is:

$$u_1 = \frac{p_1}{q_1 +} \frac{p_2}{q_2 +} \dots \frac{p_n}{q_n + u_{n+1}} = \frac{A_n + f_{n+1} A_{n-1}}{B_n + f_{n+1} B_{n-1}} \quad n = 1, 2, 3 \dots (10)$$

where the functions $A_i$and $B_i$satisfy the recursion (10).

Division Algorithm

Any pair $b_0 > b_1$ of positive integers generates a decreasing sequence $b_0 > b_1 > b_2 \cdot \cdot \cdot$ in the set N of all positive integers

$$b_0 = a_0 b_0 + b_1,$$

$$b_1 = a_1 b_2 + b_2, \qquad (11)$$

$$b_3 = a_2 b_2 + b_3,$$

$$\vdots$$

$$b_{n-2} = a_{n-2} b_{n-1} + b_n,$$

$$b_{n-1} = a_{n-1} b_n ,$$

Here $a_i \in N, i = 0, 1, \dots$. Any decreasing sequence in N is finite. So, there exists n ∈ N such that $b_{n-1} = a_{n-1} b_n$, hence the algorithm stops at this step. Reading the equations in (11) from the up to the equation $b_{n-2} = a_{n-2} b_{n-1} + b_n$, preceding the last equation $b_{n-1} = a_{n-1} b_n$, we get that any common divisor of b_0 and b_1 divides b_n. Reading the same equations from the down to the up, we get that b_n is a common divisor of b_0 and b_1. Thus, b_n is the greatest common divisor $d = (b_0, b_1)$ for $b_0$and b_1. This is the standard form of Euclidean algorithm that provides a basis for multiplicative Number Theory. To discuss the role played by the coefficients a_m in (11), we consider (11) as a system of linear algebraic equations with integer coefficients a_0, a_1, a,.... Excluding unknowns b_m from (11), we obtain $\frac{b_{m-1}}{b_m} = a_{m-1} + \frac{1}{b_m / b_{m+1}} + 1$, $m = 1, 2, \dots$, which clearly yields the development of $\frac{b_0}{b_1}$ into a finite regular continued fraction (Khrushchec, 2008; Khrushchec, 2005). Consequently, the Euclidean algorithm has a close relationship with continued fractions (Vinogradov, 2016). The sequence of equations can be written in the form

$$\frac{p}{q} = a_0 + \frac{b_0}{b}$$

$$\frac{b}{b_0} = a_1 + \frac{b_1}{b_0},$$

$$\frac{b_0}{b_1} = a_2 + \frac{b_2}{b_1},$$

$$\vdots$$

$$\frac{b_{n-2}}{b_{n-1}} = a_n + \frac{b_n}{b_{n-1}}$$

$$\vdots$$

$$\frac{b_{N-2}}{b_{N-1}} = a_N \ .$$

The last term on the right-hand side always equals the inverse of the left-hand side of the next equation. Hence, the first two equations may be combined to form

$$\frac{p}{q} = a_0 + \frac{1}{a_1 + \frac{b_0}{b_1}}$$

The final ratio of remainders $\frac{b_n}{b_{n-1}}$ can always be replaced by using the next equation in the series, up to the final equation. Then the result is a continued fraction

$$\frac{q}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_N}}}}} = [a_0; a_1, a_2, .., a_N] \qquad (12)$$

This illustrates that any rational number equals the value of a regular continued fraction (12), where a_0 is an integer ($a_0 \in$ Z) and a_1, a_2, a_{n-1} are positive integers. The feature of such a representation compared with common decimal or dyadic representations is that it is global and does not reflect properties of the base. Hence, the continuum R of real numbers can be parameterized via a sequence of integer parameters $\{a_m\}_{m\geq 0}$ restricted to $a_0 \in Z, a_m \in N \; if m \geq 1$ (Khrushchec, 2008; Krushchec, 2005).

Example 4. The continued fraction $[2, 3, 1, 2, 3] = \frac{84}{37}$ has convergence

$$[2, 3, 1, 2, 3] = \frac{84}{37} \ , [2;\ 3;\ 1;\ 2] = \frac{25}{11},\ [2;\ 3;\ 1] = \frac{9}{4},\ [2;\ 3] = \frac{7}{3}$$

Let us compute the continued fraction. The Euclidean algorithm on the left gives the continued fraction $[2, 3, 1, 2, 3] = \frac{84}{37}$. The algorithm on the right gives the even continued fraction $[2, 4, -4, 2, -2] = \frac{84}{37}$.

$\mathbf{84 = 2.37 + 10}$	$\mathbf{84 = 2.37 + 10}$
$\mathbf{37 = 3.10 + 7}$	$\mathbf{37 = 4.10 + (-3)}$
$\mathbf{10 = 1.7 + 3}$	$\mathbf{10 = (-4)(-3) + (-2)}$
$\mathbf{7 = 2.3 + 1}$	$\mathbf{-3 = 2(-2) + 1}$
$\mathbf{3 = 3.1}$	$\mathbf{-2 = (-2)1}$

An Application

The Euclidean algorithm also has a relationship to the binary tree on the rational numbers. Since it can be employed to order the set of all positive rational numbers into a binary search tree, it can see this division algorithm on the level of the tree graph, and its connected components (subgraphs) as illustrated in Figure (5). In this figure, the numbers represent the number of vertices of the binary tree from the root to the end (leaves). The corresponding binary trees are the following.

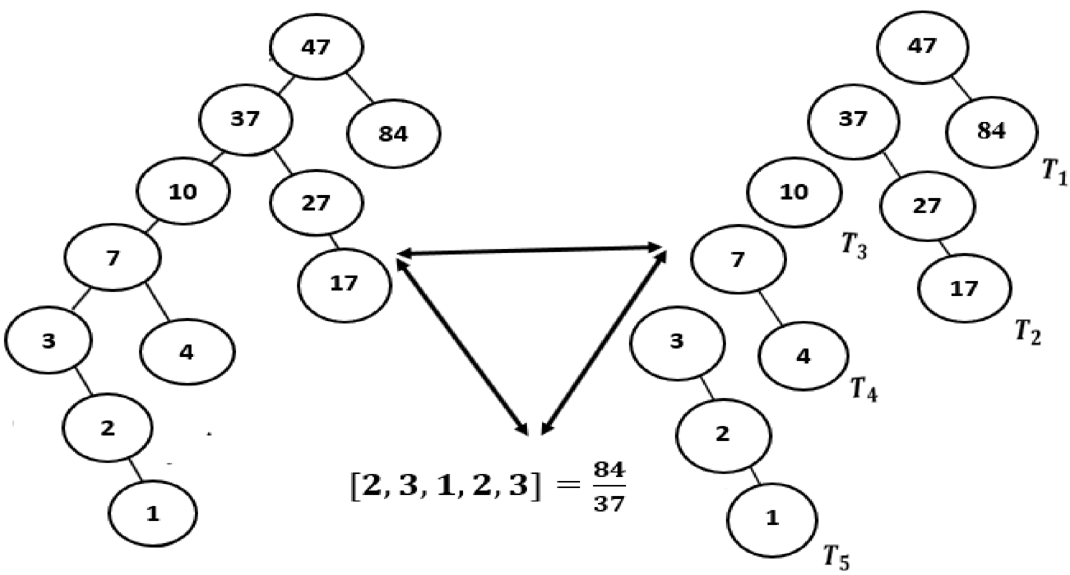


Figure (5): Binary Tree and Sub Binary Trees Continued Fraction.

We point out that the remainders can also be realized as numbers of connected components of binary tree if one starts counting at the first vertex (root) until the last vertex (leaves) of the binary tree, and the division algorithm can be seen as a sequence of identities of binary tree as follows.

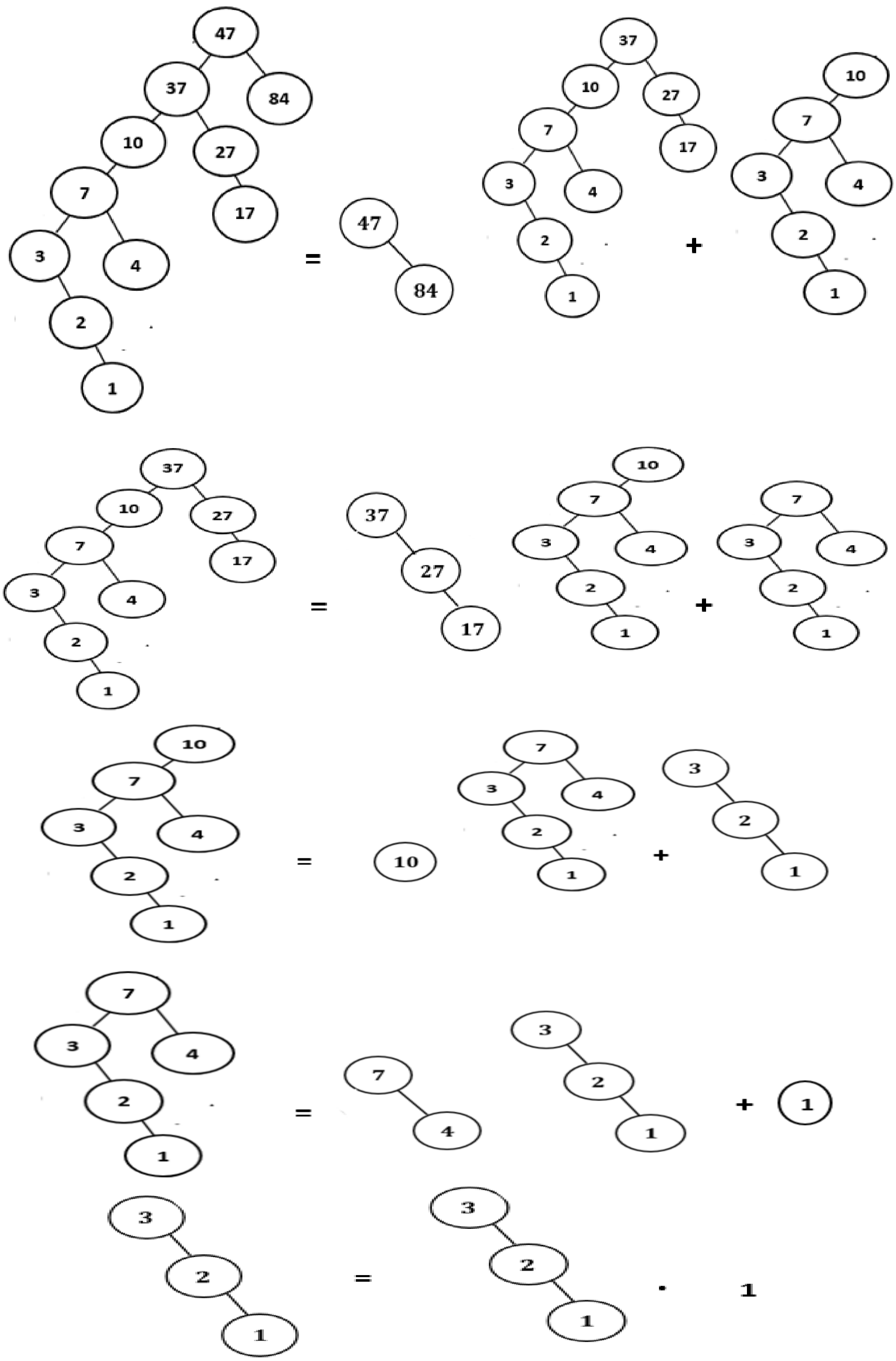


Figure (6): The Division Algorithm in Terms of Sub Binary Tress of Binary Tree.

Let’s go back to our example 1 of the continued fraction $[2,3,4] = \frac{30}{13}$.

To compute the continued fraction starting from the rational number $\frac{30}{13}$, we used the following division algorithm.

$$\mathbf{30 = 2 \cdot 13 + 4}$$

$$\mathbf{13 = 3 \cdot 4 + 1}$$

$$\mathbf{4 = 4 \cdot 1.}$$

The binary tree on the rational numbers used division algorithm is

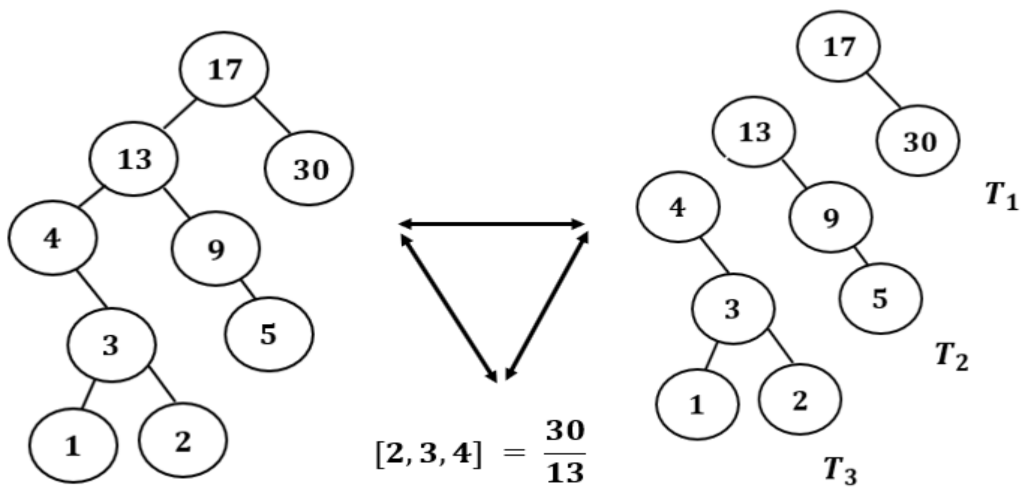
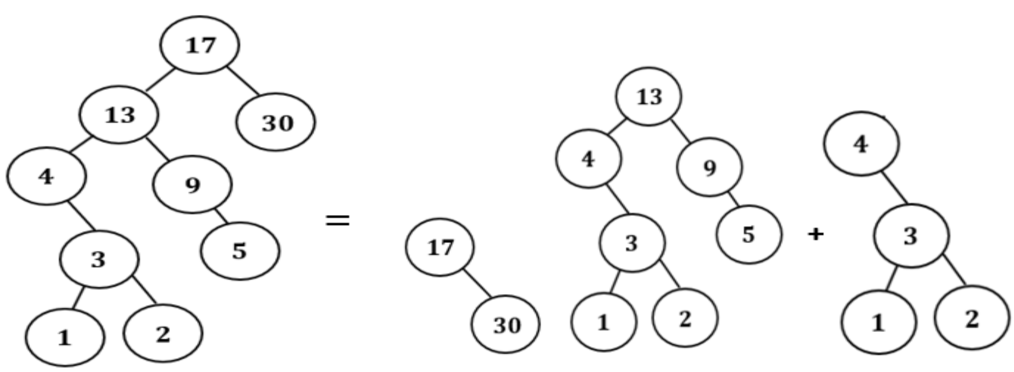


Figure (7): Binary Tree and Sub Binary Tree Continued Fraction.

The division algorithm can be seen as a sequence of identities of binary tree as follows:



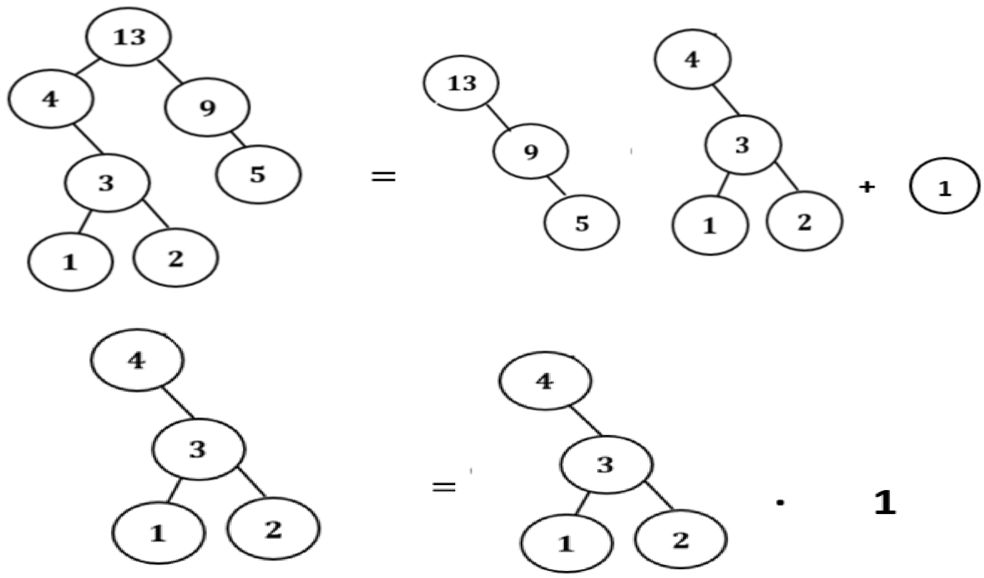


Figure (8): The Division Algorithm in Terms of Sub Binary Tress of Binary Tree.

Conclusions

We have seen that there is great combinatorial between the binary trees and the continued fractions, which can be understood by viewing such continued fractions as paths in the binary tree. Using the possible tree graphs makes interesting the combinatorial interpretation realization of convergent of the continued fractions with the binary trees and the number of connected components of the binary trees on rational numbers. In addition, this realization reflects the convergence of continued fractions with the topological index Z and the Euclidean division algorithm as well.

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