INCLUSION PROPERTIES FOR SUBCLASSES OF ANALYTIK FUNCTION BASED ON GENERALIZED DERIVATIVE OPERATOR

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Abstract

The main objective of this study was to study some inclusion properties for subclasses of analytic univalent functions introduced by using the technique of subordinations on generalized derivative operator; $D^{k,\alpha,\beta}_{\lambda 1,\lambda 2}$ areinvestigated.

Keywords: differential operator; sufficient conditions; subordinations.

Introduction

Inclusion properties for generalized differential operator have been studied earlier by many different authors. Darus and Ibrahem (2009) have studied some inclusion properties for generalized derivative operator and we are using their methods to define new subclass and investigate the various inclusion properties of these subclasses.

Let A be the subclass of H consisting of functions that take the form given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and use the derivative operator which was provided by Lsheli and Alargat (2021):

$$\boldsymbol{D^{k,\alpha,\beta}_{\lambda_1\lambda_2} f(z) = Z + \sum_{n=2}^{\infty}\left[\frac{n^{\alpha} + (n-1)n^{\alpha}\lambda_1}{n^{\beta} + (n-1)n^{\beta}\lambda_2}\right]^{k} a_n z^n} \qquad (1)$$

Note that in Lemma 1.1 (Darus & Ibrahim, 2009) Let $f \in A$. Then

(i) $D^{0,\alpha,\beta}_{\lambda_1,\lambda_2} f(z) = f(z)$.

(ii) $D^{1,1,0}_{0,0} f(z) = zf'(z)$.

(iii) $D^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2} f(z) = (1-\lambda_1)[D^{k.\alpha.\beta}_{\lambda_1,\lambda_2} f(z) * \varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)] + \lambda_1 z[D^{k.\alpha.\beta}_{\lambda_1,\lambda_2} f(z) * \varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)]'$.

Let F and G be analytic functions in the unit disk U. The function F is subordinate to G, written $F \prec G$, if G is univalent, $F(0)=G(0)$ and $F(U)\subset G(U)$. In general, given two functions F and G, which are analytic in U the function F, is said to be subordinate to $G(z)$ in U if there exists a function *h*, analytic in U with $h(0)=0$ *and* $|h(z)|<1$ *for all* $z\in U$

such that

$$F(z)=G(h(z)) \text{ for all } z\in U.$$

Let $\phi:\mathbb{C}^2\to\mathbb{C}$ and h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z),zp'(z))\prec h(z)$, then p is called a solution of the differential subordination. The univalent function q is called a dominant of solutions of the differential subordination, $p\prec q$. If p and $\phi(p(z),zp'(z))$ are univalent in U and satisfy the differential superordination $h(z)\prec\phi(p(z),zp'(z))$, then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q\prec p$. Let Φ be an analytic function in a domain containing $f(U),\Phi(0)=0$ and $\Phi'(0)>0$.

Let N be classes of all functions ϕ which are analytic and univalent in U for which $\phi(U)$ is convex wit $\phi(0)=1$ and $R\{\phi\}>0$ *for* $z\in U$. Making use of the principle of subordination between analytic functions, we introduce subclasses $S^*(\mu;\phi)$ and $C(\mu;\phi)$ of the class A for $\mu\ge 0$ and $\phi\in\mathrm{N}$ which are defined by

$$S^*(\mu;\phi)=\left\{f\in A:\frac{1}{1-\mu}\left(\frac{zf'(z)}{f(z)}-\mu\right)\prec\phi(z),z\in U\right\}$$

$$C^*(\mu;\phi)=\left\{f\in A:\frac{1}{1-\mu}\left(1+\frac{zf''(z)}{f'(z)}-\mu\right)\prec\phi(z),z\in U\right\}$$

and

$$Q^*(\mu,\beta;\phi,\psi)=\{f\in A:\exists g\in S^*(\mu;\phi)\ s.t.\ \frac{1}{1-\beta}\left(1+\frac{zf'(z)}{g(z)}-\beta\right)\prec\psi(z),z\in U\}$$

Next by using the operator $D_{\lambda_1,\lambda_2}^{k.\alpha,\beta} f(z)$, we introduce the following classes of analytic functions

$$S_{\lambda_1,\lambda_2}^{k.\alpha.\beta}(\mu;\phi)=\{f \in A, D_{\lambda_1,\lambda_2}^{k.\alpha.\beta} f(z)\in S^*(\mu;\phi)\}$$

and

$$C_{\lambda_1,\lambda_2}^{k.\alpha.\beta}(\mu;\phi)=\{f \in A, D_{\lambda_1,\lambda_2}^{k.\alpha.\beta} f(z)\in C^*(\mu;\phi)\}$$

and

$$Q_{\lambda_1,\lambda_2}^{k.\alpha.\beta}(\mu,\beta;\phi,\psi)=\{f \in A, D_{\lambda_1,\lambda_2}^{k.\alpha.\beta} f(z)\in Q^*(\mu,\beta;\phi,\psi)\}.$$

$$f(z)\in C^*(\mu;\phi)\Leftrightarrow zf'(z)\in S^*(\mu;\phi).$$

In this paper, we investigate several inclusion properties of classes $S_{\lambda_1,\lambda_2}^{k,\alpha,\beta}(\mu;\phi), C_{\lambda_1,\lambda_2}^{k,\alpha,\beta}(\mu;\phi)$ and $Q_{\lambda_1,\lambda_2}^{k,\alpha,\beta}(\mu,\beta;\phi,\psi)$ associated with the differential operator $D_{\lambda_1,\lambda_2}^{k,\alpha,\beta} f(z)$. For this purpose, the following results are needed in the sequel.

Lemma 1.2 (Eenigenburg, Miller, Mocanu & Reade, 1983) Let ϕ be convex univalent in U with $\phi(0)=1$ and $R\{k\phi(z)+v\}>0$ for $k,v\in \mathbb{C}$. If p is analytic in U with $p(0)=1$, then

$$p(z)+\frac{zp'(z)}{kp(z)+v}\prec \phi(z), \quad z\in U,$$

implies that

$$p(z)\prec \phi(z), \quad z\in U.$$

Lemma 1.3 (Miller & Mocanu, 1981) Let ϕ be convex univalent in U and ω be analytic in U with $R\{\omega\}\geq 0$. If p is analytic in U with $p(0)=\phi(0)$, then.

$$p(z)+\omega(z)zp'(z)\prec \phi(z)$$

implies that

$$p(z)\prec \phi(z), \; z\in U.$$

Inclusion Properties

In this section, we establish inclusion properties of the subclasses $S_{\lambda_1,\lambda_2}^{k,\alpha,\beta}(\mu;\phi)$, $C_{\lambda_1,\lambda_2}^{k,\alpha,\beta}(\mu;\phi)$, and $Q_{\lambda_1,\lambda_2}^{k,\alpha,\beta}(\mu,\beta;\phi,\psi)$. In view of Lemma 1.3, we prove the following result:

Theorem 2.1 Let $k\in\mathrm{N}_0,\ \beta\geq\alpha\geq 0$ and $\lambda_2\geq\lambda_1\geq 0.$ Then for $\mu\geq 0$ and $\phi\in\mathrm{N}$

$$S_{\lambda_1,\lambda_2}^{k+1,\alpha,\beta}(\mu;\phi)\subset S_{\lambda_1,\lambda_2}^{k,\alpha,\beta}(\mu;\phi).$$

Proof: To show that $S_{\lambda_1,\lambda_2}^{k+1,\alpha,\beta}(\mu;\phi)\subset S_{\lambda_1,\lambda_2}^{k,\alpha,\beta}(\mu;\phi)$. Let $f\in S_{\lambda_1,\lambda_2}^{k+1,\alpha,\beta}(\mu;\phi)$ and set

$$p(z)=\frac{1}{1-\mu}\left[\frac{z\left(D_{\lambda_1,\lambda_2}^{k.\alpha.\beta}f(z)*\varphi_{0,\lambda_2}^{1.\alpha.\beta}(z)\right)'}{D_{\lambda_1,\lambda_2}^{k.\alpha.\beta}f(z)*\varphi_{0,\lambda_2}^{1.\alpha.\beta}(z)}-\mu\right], \qquad \textbf{(2)}$$

where $p(z)$ is analytic in U with $p(0)=1.$ By applying Lemma 1.1 (iii) and (2), we have

$$p(z)=\frac{1}{1-\mu}\left[\frac{\frac{1}{\lambda_1}D_{\lambda_1,\lambda_2}^{k+1.\alpha,\beta}f(z)-(\frac{1}{\lambda_1}-1)\,[D_{\lambda_1,\lambda_2}^{k.\alpha,\beta}f(z)*\varphi_{0,\lambda_2}^{1.\alpha.\beta}(z)]}{D_{\lambda_1,\lambda_2}^{k.\alpha.\beta}f(z)*\varphi_{0,\lambda_2}^{1.\alpha.\beta}(z)}-\mu\right]$$

$$p(z)(1-\mu)=\left(\frac{1}{\lambda_1}\right)\frac{D_{\lambda_1,\lambda_2}^{k+1.\alpha,\beta}f(z)}{D_{\lambda_1,\lambda_2}^{k.\alpha.\beta}f(z)*\varphi_{0,\lambda_2}^{1.\alpha.\beta}(z)}-(\frac{1}{\lambda_1}-1)-\mu,$$

So

$$\left(\frac{1}{\lambda_1}\right)\frac{D_{\lambda_1,\lambda_2}^{k+1.\alpha,\beta}f(z)}{D_{\lambda_1,\lambda_2}^{k.\alpha.\beta}f(z)*\varphi_{0,\lambda_2}^{1.\alpha.\beta}(z)}=p(z)(1-\mu)+\left(\frac{1}{\lambda_1}-1\right)+\mu. \qquad \textbf{(3)}$$

Taking the logarithm differentiation on both sides of (3) and multiplying by$z$, we have

$$log\left(\frac{1}{\lambda_1}\right)\frac{D_{\lambda_1,\lambda_2}^{k+1.\alpha,\beta}f(z)}{D_{\lambda_1,\lambda_2}^{k.\alpha.\beta}f(z)*\varphi_{0,\lambda_2}^{1.\alpha.\beta}(z)}=log\ p(z)(1-\mu)+\left(\frac{1}{\lambda_1}-1\right)+\mu.$$

Left side:

$$log\left(\frac{1}{\lambda_1}\right)\frac{D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)}{D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)}=\frac{D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)}{\left(\frac{1}{\lambda_1}\right)D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)}$$

$$\times\frac{D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)\left(\frac{1}{\lambda_1}\right)[D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)]'-\left(\frac{1}{\lambda_1}\right)D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)[D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)]'}{[D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)]^2}$$

$$=\frac{D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)\left[D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)\right]'}{D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)(D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z))}-\frac{D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)[D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)]'}{D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)(D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z))}$$

$$=\frac{\left[D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)\right]'}{D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)}-\frac{[D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)]'}{D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)}.$$

Right side:

$$\log p(z)(1-\mu)+\left(\frac{1}{\lambda_1}-1\right)+\mu=\frac{p'(z)(1-\mu)}{p(z)(1-\mu)+\left(\frac{1}{\lambda_1}-1\right)+\mu}.$$

Now by equalizing both sides we have and multiplying them by z we have

$$\frac{\left[D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)\right]'}{D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)}-\frac{[D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)]'}{D^{k.\alpha.\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)}=\frac{p'(z)(1-\mu)}{p(z)(1-\mu)+\left(\frac{1}{\lambda_1}-1\right)+\mu},$$

$$\frac{1}{1-\mu}\left[\frac{\left[D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)\right]'}{D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)}-p(z)+\mu\right]=\frac{zp'(z)}{p(z)(1-\mu)+\left(\frac{1}{\lambda_1}-1\right)+\mu},$$

$$\frac{1}{1-\mu}\left[\frac{\left[D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)\right]'}{D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)}-p(z)+\mu\right]=\frac{zp'(z)}{p(z)(1-\mu)+\left(\frac{1}{\lambda_1}-1\right)+\mu}, \qquad \textbf{(4)}$$

Applying Lemma 1.2 to (3), it follows that $p(z)\prec\phi,$ that is $f\in S^{k,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;\phi).$

Theorem. 2.2 Let $k \in \mathbb{N}_0$, $\beta \geq \alpha \geq 0$, and $\lambda_2 \geq \lambda_1 \geq 0$. Then for $\mu \geq 0$ and $\phi \in \mathrm{N}$

$$C^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;\phi) \subset C^{k,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;\phi).$$

Proof: By applying (1) and Theorem 2.1, we observe that

$$f \in C^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;\phi) \Leftrightarrow D^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2} f(z) \in C^*(\mu;\phi)$$

$$\Leftrightarrow z[D^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2} f(z)]' \in S^*(\mu;\phi)$$

$$\Leftrightarrow D^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2}[zf(z)]' \in S^*(\mu;\phi)$$

$$\Leftrightarrow [zf(z)]' \in S^{k,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;\phi)$$

$$\Leftrightarrow D^{k,\alpha,\beta}_{\lambda_1,\lambda_2}[zf(z)]' \in S^*(\mu;\phi)$$

$$\Leftrightarrow z[D^{k,\alpha,\beta}_{\lambda_1,\lambda_2} f(z)]' \in S^*(\mu;\phi)$$

$$\Leftrightarrow D^{k,\alpha,\beta}_{\lambda_1,\lambda_2} f(z) \in C^*(\mu;\phi)$$

$$\Leftrightarrow f \in C^{k,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;\phi).$$

Therefore, $C^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;\phi) \subset C^{k,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;\phi)$. This completes the proof.

Taking $\phi(z) = \dfrac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in U$ in Theorem 2.1 and Theorem 2.2, we have the following result.

Corollary 2.3 Let $k \in \mathbb{N}_0$, $\beta \geq \alpha \geq 0$, and $\lambda_2 \geq \lambda_1 \geq 0$. Then for $\mu \geq 0$ and $-1 \leq B < A \leq 1$

$$S^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;A,B) \subset S^{k,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;A,B)$$

and

$$C^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;A,B) \subset C^{k,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;A,B).$$

Next by Lemma 1.3, the following inclusion relation for the class are obtained $Q^{k,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu,\beta;\phi)$.

Theorem 2.4 Let $k\in \mathrm{N}_0$, $\beta\geq\alpha\geq 0,$ and $\lambda_2\geq\lambda_1\geq 0.$ Then for $\mu,\beta\geq 0$ and $\phi,\psi\in \mathrm{N}$

$$Q^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu,\beta;\phi)\subset Q^{k,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu,\beta;\phi).$$

Proof: Let $f\in Q^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu,\beta;\phi)$. Then there exists $g\in S^{k+1,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;\phi)$ such that

$$\frac{1}{1-\beta}\left(\frac{z\left[D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)\right]'}{D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}g(z)}-\beta\right)\prec\psi(z),\quad z\in U.$$

Set

$$p(z)=\frac{1}{1-\beta}\left(\frac{z\left[D^{k.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)\right]'}{D^{k.\alpha,\beta}_{\lambda_1,\lambda_2}g(z)}-\beta\right), \qquad \textbf{(5)}$$

where $p(z)$ is analytic in U with $p(0)=1$. Applying Lemma 1.1 (iii) and (5), we obtain

$$\left[(1-\beta)p(z)+\beta\right]D^{k.\alpha,\beta}_{\lambda_1,\lambda_2}g(z)$$

$$=\left(\frac{1}{\lambda_1}\right)D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)-(\frac{1}{\lambda_1}-1)D^{k.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z).$$

By differentiating on both sides and multiplying by z, we have

$$(1-\beta)p'(z)z\,D^{k.\alpha,\beta}_{\lambda_1,\lambda_2}g(z)+\beta z[D^{k.\alpha,\beta}_{\lambda_1,\lambda_2}g(z)]'$$

$$=\left(\frac{1}{\lambda_1}\right)z[D^{k+1.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)]'-(\frac{1}{\lambda_1}-1)z\,[D^{k.\alpha,\beta}_{\lambda_1,\lambda_2}f(z)*\varphi^{1.\alpha.\beta}_{0,\lambda_2}(z)]'. \qquad \textbf{(6)}$$

Since $g\in S^{k,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;\phi)$.

in view of Theorem 2.1, $g\in S^{k,\alpha,\beta}_{\lambda_1,\lambda_2}(\mu;\phi)$.

Let

$$q(z)=\frac{1}{1-\mu}\left[\frac{z\left[D_{\lambda_1,\lambda_2}^{k.\alpha,\beta}g(z)\right]'}{D_{\lambda_1,\lambda_2}^{k.\alpha,\beta}g(z)}-\mu\right], \qquad \textbf{(7)}$$

where q is analytic in U with $q(0)=1.$ Applying Lemma 1.1 (iii) and (7), we obtain

$$q(z)=\frac{1}{1-\mu}\left[\frac{\frac{1}{\lambda_1}D_{\lambda_1,\lambda_2}^{k+1.\alpha,\beta}g(z)-(\frac{1}{\lambda_1}-1)D_{\lambda_1,\lambda_2}^{k.\alpha,\beta}g(z)}{D_{\lambda_1,\lambda_2}^{k.\alpha.\beta}g(z)}-\mu\right],$$

$$q(z)(1-\mu)=\left(\frac{1}{\lambda_1}\right)\frac{D_{\lambda_1,\lambda_2}^{k+1.\alpha,\beta}g(z)}{D_{\lambda_1,\lambda_2}^{k.\alpha.\beta}g(z)}-(\frac{1}{\lambda_1}-1)-\mu,$$

So

$$\left(\frac{1}{\lambda_1}\right)\frac{D_{\lambda_1,\lambda_2}^{k+1.\alpha,\beta}g(z)}{D_{\lambda_1,\lambda_2}^{k.\alpha.\beta}g(z)}=p(z)(1-\mu)+\left(\frac{1}{\lambda_1}-1\right)+\mu \qquad \textbf{(8)}$$

From (6) and (8), we have

$$\frac{1}{1-\beta}\left(\frac{z\left[D_{\lambda_1,\lambda_2}^{k+1.\alpha,\beta}f(z)\right]'}{D_{\lambda_1,\lambda_2}^{k+1.\alpha,\beta}g(z)}-\beta\right)=p(z)+\frac{zp'(z)}{q(z)(1-\mu)+\left(\frac{1}{\lambda_1}-1\right)+\mu}.$$

Since $q\prec\phi$ and $R\{q(z)(1-\mu)+\left(\frac{1}{\lambda_1}-1\right)+\mu>0,$ by Lemma 1.3, we have $p\prec\psi$ so that $f\in Q_{\lambda_1,\lambda_2}^{k,\alpha,\beta}(\mu,\beta;\phi,\psi).$

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