THE FEKETE-SZEGÖ PROBLEM FOR SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH DIFFERENTIAL OPERATOR

Anessa Oshah[1*], Salma Faraj[2], and Maslina Darus[3]

[1,2] Department of Mathematics, Faculty of science, Sabratha University
[3] School of Mathematical Sciences, Faculty of Sciences and Technology, University Kebangsaan Malaysia.
[*] anessa.oshah@yahoo.com

Abstract

The aim of this paper is to determine the Fekete-Szegö inequality for the class $\mathcal{M}^{n,m}_{\lambda_1,\lambda_2,\ell,d}(\emptyset)$ of normalized analytic functions $f(z)$ defined on the open unit disk for which $\frac{z\left(D^{n,m}_{\lambda_1,\lambda_2,\ell,d} f\ (z)\right)'}{D^{n,m}_{\lambda_1,\lambda_2,\ell,d} f\ (z)}$ lies in a region starlike with respect to 1 and symmetric with respect to the real axis by using the differential operator $D^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)$ given by authors Oshah and Darus. As a special case of this result, Fekete-Szegö inequality for a class of functions that is defined by fractional derivatives is to be obtained too.

Keywords: analytic function; starlike function; subordination; Fekete-Szegö inequality; differential operator.

Introduction

Let Adenote the class of functions of the form

$$\boldsymbol{f(z) = z + \sum_{k=2}^{\infty} a_k z^k, (z \in \mathbb{U}),} \qquad (\mathbf{1})$$

which is analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Also, let Sbe the family of functions $f \in A$, which are univalent. If the functions $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$; we say that $f(z)$ is subordinate to $g(z)$, which is written as

$$\boldsymbol{f \prec g \text{ or } f(z) \prec g(z)}$$

If a Schwarz function exists, $w(z)$; which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$ such that $f(z) = g\big(w(z)\big)$; $z \in \mathbb{U}$. Further, let P denote the class of analytic functions in U such that $h(z) = 1 + p_1 z + p_2 z^2 + \cdots$, $h(0) = 1$ and $\Re e h(z) > 0$, $h(z) = \frac{1+w(z)}{1-w(z)}$, for some $z \in \mathbb{U}$.

For two analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ their convolution (orHadamard product) is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $\emptyset \in$ P, where $\emptyset\ (z)$ is an analytic function with positive real part on A with $\emptyset\ (0) = 1, \emptyset'(0) > 0$, and let $S^*(\emptyset\ )$ be the class of functions in $f \in S$ for which

$$\frac{zf'\ (z)}{f\ (z)} \prec \emptyset\ (z), \quad (z \in \mathbb{U}), \qquad (2)$$

and $C(\emptyset)$ be the class of functions in $f \in S$ for which

$$1 + \frac{zf''\ (z)}{f'\ (z)} \prec \emptyset\ (z), \quad (z \in \mathbb{U}), \qquad (3)$$

where $\prec$ denotes the subordination between analytic functions.

These classes were defined and studied by Ma and Minda (1994). They have obtained the Fekete-Szegö inequality for the functions in the class $C(\emptyset)$. Since $f \in C(\emptyset)$ if and only if $zf'\ (z) \in S^*(\emptyset\ )$, we get the Fekete-Szegö inequality for functions in the class $S^*(\emptyset)$. For a brief history of the Fekete-Szegö problem for the class of starlike, convex and close to convex functions have been mentioned by Mohammed and Darus (2010), Srivastava *et al.* (2001), Darus (2002), Al-Abbadi and Darus (2011), Ravichandran *et al.* (2004) and Al-Shaqsi and Darus (2008), Salma and Darus(2011) .

Oshah and Darus (2014) introduced a differential operator $D_{\lambda_1,\lambda_2,\ell,d}^{n,m}\ f(z)$ by:

$$D_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\ell(1 + (\lambda_1 + \lambda_2)(k-1)) + d}{\ell(1 + \lambda_2(k-1)) + d} \right)^m C(n,k) a_k z^k, \qquad (4)$$

where

$n, m, d \in \mathbb{N}_0, \lambda_1 \geq \lambda_2 \geq 0, \ell \geq 0$, and $C(n,k) = \binom{k+n-1}{n}, k = 2,3,4, \ldots$.

Using the operator $D_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)$. Let us define the class proposed as follows:

Definition 1:

Let $\emptyset \in$ P be a univalent starlike function with respect to 1, which maps the unit disc $\mathbb{U}$ onto a region in the right half plane and symmetric with respect to the real axis, $\emptyset\ (0) = 1, \emptyset'(0) > 0$. A function $f \in A$ is in the class $\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m}(\emptyset)$

$$\frac{z\left(D^{n,m}_{\lambda_1,\lambda_2,\ell,d}f\,(z)\right)'}{D^{n,m}_{\lambda_1,\lambda_2,\ell,d}f\,(z)} \prec \emptyset(z). \qquad (5)$$

To prove our main results, the following lemma is required (Ma & Minda 1994).

Lemma 1:

If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$, it is an analytic function with positive real part in $\mathbb{U}$ then

$$\left|c_2 - v c_1^2\right| \le \begin{cases} -4v + 2 & \text{if} \quad v \le 0 \\ 2 & \text{if} \quad 0 \le v \le 0 \\ 4v - 2 & \text{if} \quad v \ge 1 \end{cases}.$$

When $v < 0$or $v > 1$, the equality holds if and only if $p_1(z)$ is $(1 + z)/(1 - z)$or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p_1(z)$ is $(1 + z^2)/(1 - z^2)$or one of its rotations. If $v = 0$,the equality holds if and only if

$$p_1(z) = \left(\frac{1+\alpha}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\alpha}{2}\right)\frac{1-z}{1+z} \quad (0 \le \alpha \le 1; z \in \mathbb{U})$$

or one of its rotations. If $v = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$. Also the above upper bound is sharp, it can be improved as follows

when $0 < v < 1$:

$$\left|c_2 - v c_1^2\right| + v|c_1|^2 \le 2, \quad \left(0 < v \le \frac{1}{2}\right),$$

and $|c_2 - v c_1^2| + (1 - v)|c_1|^2 \le 2, \quad \left(\frac{1}{2} < v \le 1\right)$.

In the present paper, we obtained the Fekete-Szegö inequality for functions in a more general class $\mathcal{M}^{n,m}_{\lambda_1,\lambda_2,\ell,d}(\emptyset)$ of functions, which we defined above. In addition, we applied the results to certain functions defined through convolution (or the Hadamard product) particularly we considered a class $\mathcal{M}^{n,m,\gamma}_{\lambda_1,\lambda_2,\ell,d}(\emptyset)$ of functions defined by fractional derivatives. The motivation of this paper is to generalize the Fekete-Szegö inequalities proved by Srivastava and Mishra (2000) for functions in the class $\mathcal{M}^{n,m,\gamma}_{\lambda_1,\lambda_2,\ell,d}(\emptyset)$.

Main Results

Our main result is the following:

Theorem1

Let $\emptyset(z)$ be an analytic function with positive $\emptyset(\mathrm{z}) = 1 + \mathrm{B}_1\mathrm{z} + \mathrm{B}_2\mathrm{z}^2 + \ldots\ldots\ldots$ If f (z) is given by (1) and belongs to $\mathcal{M}^{n,m,\gamma}_{\lambda_1,\lambda_2,\ell,d}(\emptyset)$ then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{B_2}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m - \frac{\mu B_1^2}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m} \\ \quad + \frac{B_1^2}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m if \mu \le \sigma_1; \\ \frac{B_1}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m if \sigma_1 \le \mu \le \sigma_2; \\ -\frac{B_2}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m + \frac{\mu B_1^2}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m} \\ -\frac{B_1^2}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m \quad if \mu \ge \sigma_2, \end{cases} \quad (6)$$

where

$$\sigma_1 = \frac{(n+1)^2}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m \left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m} \left\{\frac{(B_2-B_1)+{B_1}^2}{{B_1}^2}\right\}, (7)$$

$$\sigma_2 = \frac{(n+1)^2}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m \left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m} \left\{\frac{(B_2+B_1)+{B_1}^2}{{B_1}^2}\right\}. (8)$$

Proof

For $f(z) \in \mathcal{M}^{n,m}_{\lambda_1,\lambda_2,\ell,d}(\emptyset)$, let

$$p(z) = \frac{z(D^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z))'}{D^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)} = 1 + b_1 z + b_2 z^2 + \cdots. (9)$$

Substituting (4) in (9) and comparing the coefficients of z^2 and z^3 on both sides in equation (9), we have

$$(n+1)\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^m a_2 = b_1,$$

and

$$(n+2)(n+1)\left(\frac{\ell+2\ell(\lambda_1+\lambda_2)+d}{\ell+2\ell\lambda_2+d}\right)^m a_3 = (n+1)^2\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m} {a_2}^2 + b_2, \quad (10)$$

We wanted to find out the values for b_1 and b_2. Since $\emptyset(z)$ is univalent and $p \prec \emptyset$, the function $p_1(z) = \frac{1+\emptyset^{-1}(p(z))}{1-\emptyset^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \cdots$, is analytic and has a positive real part in U. Thus, we have

$$\boldsymbol{p(z)} = \emptyset\left(\frac{\boldsymbol{p_1(z) - 1}}{\boldsymbol{p_1(z) + 1}}\right). \qquad (\mathbf{11})$$

From the equations (9) and (11), we obtain

$$\mathbf{1} + \boldsymbol{b_1 z} + \boldsymbol{b_2 z^2} + \cdots = \emptyset\left(\frac{\boldsymbol{c_1 z} + \boldsymbol{c_2 z^2} + \cdots}{\mathbf{2} + \boldsymbol{c_1 z} + \boldsymbol{c_2 z^2} + \cdots}\right)$$

$$= \emptyset\left[\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{c_1 z} + \frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{c_2} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{c_1}^{\mathbf{2}}\right)\boldsymbol{z^2} + \cdots\right]$$

$$= \mathbf{1} + \boldsymbol{B_1}\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{c_1 z} + \boldsymbol{B_1}\frac{\mathbf{1}}{\mathbf{2}}\left(\boldsymbol{c_2} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{c_1}^{\mathbf{2}}\right)\boldsymbol{z^2} + \cdots + \boldsymbol{B_2}\frac{\mathbf{1}}{\mathbf{4}}\boldsymbol{c_1}^{\mathbf{2}}\boldsymbol{z^2} + \cdots,$$

and this implies

$b_1 = \frac{1}{2}B_1 c_1$ and $b_2 = \frac{1}{2}B_1\left(c_2 - \frac{1}{2}{c_1}^2\right) + \frac{1}{4}B_2{c_1}^2$.

Therefore, we have

$$\boldsymbol{a_3 - \mu a_2^2} = \frac{\boldsymbol{B_1}}{\mathbf{2}(\boldsymbol{n}+\mathbf{2})(\boldsymbol{n}+\mathbf{1})}\left(\frac{\boldsymbol{\ell} + \mathbf{2}\boldsymbol{\ell\lambda_2} + \boldsymbol{d}}{\boldsymbol{\ell} + \mathbf{2}\boldsymbol{\ell}(\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2}) + \boldsymbol{d}}\right)^{\boldsymbol{m}}\left\{\boldsymbol{c_2} - \boldsymbol{c_1}^{\mathbf{2}}\left[\frac{\mathbf{1}}{\mathbf{2}}\left(\mathbf{1} - \frac{\boldsymbol{B_2}}{\boldsymbol{B_1}} + \frac{\boldsymbol{B_1}}{(\boldsymbol{n}+\mathbf{1})^{\mathbf{2}}}\left(\frac{\boldsymbol{\ell} + \boldsymbol{\ell\lambda_2} + \boldsymbol{d}}{\boldsymbol{\ell} + \boldsymbol{\ell}(\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2}) + \boldsymbol{d}}\right)^{\mathbf{2}\boldsymbol{m}}\left((\boldsymbol{n}+\mathbf{2})(\boldsymbol{n}+\mathbf{1})\left(\frac{\boldsymbol{\ell} + \mathbf{2}\boldsymbol{\ell}(\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2}) + \boldsymbol{d}}{\boldsymbol{\ell} + \mathbf{2}\boldsymbol{\ell\lambda_2} + \boldsymbol{d}}\right)^{\boldsymbol{m}}\boldsymbol{\mu} - (\boldsymbol{n}+\mathbf{1})^{\mathbf{2}}\left(\frac{\boldsymbol{\ell} + \boldsymbol{\ell}(\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2}) + \boldsymbol{d}}{\boldsymbol{\ell} + \boldsymbol{\ell\lambda_2} + \boldsymbol{d}}\right)^{\mathbf{2}\boldsymbol{m}}\right)\right)\right]\right\}$$

$$= \frac{\boldsymbol{B_1}}{\mathbf{2}(\boldsymbol{n}+\mathbf{2})(\boldsymbol{n}+\mathbf{1})}\left(\frac{\boldsymbol{\ell} + \mathbf{2}\boldsymbol{\ell\lambda_2} + \boldsymbol{d}}{\boldsymbol{\ell} + \mathbf{2}\boldsymbol{\ell}(\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2}) + \boldsymbol{d}}\right)^{\boldsymbol{m}}[\boldsymbol{c_2} - \boldsymbol{v}\boldsymbol{c_1}^{\mathbf{2}}],$$

where

$$v=\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{B_1}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}\left((n+2)(n+1)\left(\frac{\ell+2\ell(\lambda_1+\lambda_2)+d}{\ell+2\ell\lambda_2+d}\right)^{m}\mu-(n+1)^2\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\right)\right].$$

If μ $\leq \sigma_1$, then by Lemma 1, we obtain

$$\left|a_3-\mu a_2^2\right|\leq\frac{B_2}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^{m}-\frac{\mu B_1^2}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}+\frac{B_1^2}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^{m}.$$

which is the first part of assertion (6).

Similarly, if μ $\geq \sigma_2$, we get

$$\left|a_3-\mu a_2^2\right|\leq-\frac{B_2}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^{m}+\frac{\mu B_1^2}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}-\frac{B_1^2}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^{m}.$$

If μ $=\sigma_1$, then equality holds if and only if

$$p_1(z)=\left(\frac{1+\alpha}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\alpha}{2}\right)\frac{1-z}{1+z},\quad (0\leq\alpha\leq 1; z\in\mathbb{U})$$

or one of its rotations.

Also, if μ $=\sigma_2$, then

$$\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{B_1}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}\left((n+2)(n+1)\mu\left(\frac{\ell+2\ell(\lambda_1+\lambda_2)+d}{\ell+2\ell\lambda_2+d}\right)^{m}-(n+1)^2\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\right)\right]=0.$$

Therefore,

$$\frac{1}{p_1(z)} = \left(\frac{1+\alpha}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\alpha}{2}\right)\frac{1-z}{1+z}, \quad (0 \le \alpha \le 1; z \in \mathbb{U}).$$

Finally, we see that

$$\left|a_3 - \mu a_2^2\right| = \frac{B_1}{2(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m \left|c_2 - c_1{}^2\left[\frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{B_1}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}\left((n+2)(n+1)\mu\left(\frac{\ell+2\ell(\lambda_1+\lambda_2)+d}{\ell+2\ell\lambda_2+d}\right)^m - (n+1)^2\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\right)\right)\right]\right|,$$

and

$$max\left|\frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{B_1}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}\left((n+2)(n+1)\mu\left(\frac{\ell+2\ell(\lambda_1+\lambda_2)+d}{\ell+2\ell\lambda_2+d}\right)^m - (n+1)^2\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\right)\right]\right| \le 1, \qquad (\sigma_1 \le \mu \le \sigma_2).$$

Therefore using Lemma 1, we get

$$\left|a_3 - \mu a_2^2\right| = \frac{B_1}{2(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m |c_1| \le \frac{B_1}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m, (\sigma_1 \le \mu \le \sigma_2).$$

If $\sigma_1 \le \mu \le \sigma_2$

$$p_1(z) = \frac{1+\alpha z^2}{1-\alpha z^2}, \ 0 \le \alpha \le 1.$$

Now result will be followed by an application of Lemma 1. To show that these bounds are sharp, we define the functions $Q_\emptyset^\delta(\delta = 2,3,\ldots)$ are defined by

$$p(z) = \frac{z\left(D_{\lambda_1,\lambda_2,\ell,d}^{n,m}\ Q_\emptyset^\delta(z)\right)'}{D_{\lambda_1,\lambda_2,\ell,d}^{n,m}\ Q_\emptyset^\delta(z)} = \emptyset\left(z^{\delta-1}\right),\ Q_\delta^\emptyset(0) = 0 = \left(Q_\delta^\emptyset(0)\right) - 1',$$

and functions F_γ and $E_\gamma(0 \le \gamma < 1)$ by

$$\frac{z\left(D^{n,m}_{\lambda_1,\lambda_2,\ell,d}F_\gamma(z)\right)'}{D^{n,m}_{\lambda_1,\lambda_2,\ell,d}F_\gamma(z)} = \emptyset\left(\frac{z(z+\gamma)}{1+\gamma z}\right), F_\gamma(0) = 0 = \left(F_\gamma(0)\right)' - 1,$$

to show that the bounds are sharp, and

$$\frac{z\left(D^{n,m}_{\lambda_1,\lambda_2,\ell,d}\ E_\gamma(z)\right)'}{D^{n,m}_{\lambda_1,\lambda_2,\ell,d}\ E_\gamma(z)} = \emptyset\left(-\frac{z(z+\gamma)}{1+\gamma z}\right),\ E_\gamma(0) = 0 = \left(\ E_\gamma(0)\right)' - 1.$$

It is obvious that, the functions of $Q^{\delta}_{\emptyset}$, F_γ, $E_\gamma \in \mathcal{M}^{n,m}_{\lambda_1,\lambda_2,\ell,d}(\emptyset)$, $Q_\emptyset \coloneqq Q^2_\emptyset$ can also be written. The equality holds for $\mu < \sigma_1$or $\mu > \sigma_2$, if and only if f is$Q_\emptyset$ or one of its rotations. The equality holds when $\sigma_1 < \mu < \sigma_2$ if and only if $f$is $Q^3_\emptyset$ or one of its rotations. The equality holds for $\mu = \sigma_1$, if and only if f is F_γ or one of its rotations. The equality holds for $\mu = \sigma_2$, if andonly if $f$is E_γ or one of its rotations.

Remark 1

Then, in view of Lemma 1, Theorem1can be improved if $\sigma_1 \le \mu \le \sigma_2$.

Now, if

$$\frac{(n+1)^2}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\left\{\frac{{B_1}^2+B_2}{{B_1}^2}\right\}$$

gives σ_3, then for the values of $\sigma_1 \le \mu \le \sigma_3$

$$\begin{aligned}
\left|a_3-\mu a_2^2\right| + \frac{(n+1)^2}{2(n+2)(n+1)B_1^2}\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\Bigg[B_1 - B_2 \\
+\frac{B_1^2}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}\Bigg((n+2)(n \\
+1)\left(\frac{\ell+2\ell(\lambda_1+\lambda_2)+d}{\ell+2\ell\lambda_2+d}\right)^m \mu \\
-(n+1)^2\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\Bigg)\Bigg]\left|a_2^2\right| \\
\le \frac{B_1}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m
\end{aligned}$$

is obtained.

Likewise

$$\left|a_3-\mu a_2^2\right|+\frac{(n+1)^2}{2(n+2)(n+1)B_1^2}\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\left[B_1+B_2-\frac{B_1^2}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}\left((n+2)(n+1)\left(\frac{\ell+2\ell(\lambda_1+\lambda_2)+d}{\ell+2\ell\lambda_2+d}\right)^{m}\mu-(n+1)^2\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\right)\right]\left|a_2^2\right| \leq \frac{B_1}{(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^{m}$$

is obtained for the values $\sigma_3 \leq \mu \leq \sigma_2$.

Proof.

For the values $\sigma_1 \leq \mu \leq \sigma_3$, we have

$$\left|a_3-\mu a_2^2\right|+(\mu-\sigma_1)|a_2|^2$$

$$=\frac{B_1}{2(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^{m}\left|c_2-vc_1^2\right|+(\mu-\sigma_1)\frac{B_1^2}{4(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}|c_1|^2$$

$$=\frac{B_1}{2(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^{m}\left|c_2-vc_1^2\right|$$

$$+\left(\mu-\frac{(n+1)^2}{(n+2)(n+1)B_1^2}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^{m}\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\left\{(B_2-B_1)+B_1^2\right\}\right)\frac{B_1^2}{4(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}|c_1|^2$$

$$=\frac{B_1}{2(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^{m}\left\{\frac{1}{2}\left|c_2-vc_1^2\right|+v|c_1|^2\right\}$$

$$\leq\frac{B_1}{2(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^{m}.$$

Similarly, for the values of $\sigma_3 \leq \mu \leq \sigma_2$ we write

$$\left|a_3-\mu a_2^2\right|+(\sigma_2-\mu)|a_2|^2$$
$$=\frac{B_1}{2(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m\left|c_2-vc_1^2\right|$$
$$+(\sigma_2-\mu)\frac{B_1^2}{4(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}|c_1|^2$$

$$=\frac{B_1}{2(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m\left|c_2-vc_1^2\right|$$
$$+\left(\frac{(n+1)^2}{(n+2)(n+1)B_1^2}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\left\{(B_2-B_1)+B_1^2\right\}-\mu\right)\frac{B_1^2}{4(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m}|c_1|^2$$

$$=\frac{B_1}{2(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m\left\{\frac{1}{2}\left|c_2-vc_1^2\right|+(1-v)|c_1|^2\right\}$$
$$\leq\frac{B_1}{2(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m$$

Therefore, Remark 1 remains true.

Applications to Functions Defined by Fractional Derivatives

For fixed $g\in A$, let $\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m,g}(\emptyset)$ be class of functions $f\in A$ for which $(f*g)\in\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m}(\emptyset)$. In order to introduce the class $\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m,\gamma}(\emptyset)$ we need the following:

Definition 2 (Owa & Srivastava, 1987) Let f be analytic function in a simply-connected region of the z- plane containing the region. The fractional derivative of f order γ is defined by

$$D_z^{\gamma}f(z)=\frac{1}{\Gamma(1-\gamma)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\gamma}}d\zeta,\qquad(0\leq\gamma<1),$$

where the multiplicity of $(z-\zeta)^{\gamma}$ is removed by requiring that $\log(z-\zeta)^{\gamma}$ is real for $(z-\zeta)^{\gamma}>0$. Using the above definition and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava (1987) introduced the operator $\Omega^{\gamma}:A\longrightarrow A$ defined by

$$(\Omega^{\gamma}f)(z)=\Gamma(2-\gamma)z^{\gamma}D_z^{\gamma}f(z),\ \ (\gamma\neq 2,3,4,\dots).$$

The class $\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m,\gamma}(\emptyset)$ consists of the functions of $f\in A$ for which $\Omega^{\gamma}f\in\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m}(\emptyset)$.

It can be noted that, when

$$\boldsymbol{g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)} z^n},$$

$\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m,\gamma}(\emptyset)$ is the special case of $\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m,g}(\emptyset)$.

Let

$$\boldsymbol{g(z) = z + \sum_{n=2}^{\infty} g_n z^n}, \qquad (\boldsymbol{g_n > 0}).$$

Since

$$\boldsymbol{D_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\ell + \ell(\lambda_1+\lambda_2)(k-1) + d}{\ell + \ell\lambda_2(k-1) + d}\right)^m a_n z^n \in \mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m,g}}(\emptyset),$$

if and only if

$$\begin{aligned}\left(\boldsymbol{D_{\lambda_1,\lambda_2,\ell,d}^{n,m} f * g}\right)(\boldsymbol{z}) \\ = \boldsymbol{z + \sum_{n=2}^{\infty} \left(\frac{\ell + \ell(\lambda_1+\lambda_2)(k-1) + d}{\ell + \ell\lambda_2(k-1) + d}\right)^m a_n g_n z^n \in \mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m}}(\emptyset).\end{aligned}$$

The estimation of the coefficient for the functions in the class $\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m,g}(\emptyset)$ is obtained from the estimation that corresponds to the functions of f in class $\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m}(\emptyset)$. If Theorem 1 is applied for operator (4), Theorem 2 is obtained after an obvious change of the parameter μ.

Theorem 2

Let g(z) = z +$\sum_{n=2}^{\infty} g_n z^n$, $(g_n > 0)$ and let the function $\emptyset(z)$ be given by $\emptyset(z) = 1 + \sum_{n=2}^{\infty} B_n z^n$. If operator given by (4) belongs to $\mathcal{M}_{\lambda_1,\lambda_2,\ell,d}^{n,m,g}(\emptyset)$, then

$$|a_3-\mu a_2^2|$$
$$\leq \begin{cases} \frac{B_2}{g_3(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m-\frac{\mu B_1^2}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m} \\ \qquad +\frac{B_1^2}{g_3(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m \text{if} \mu\leq\sigma_1; \\ \frac{B_1}{g_3(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m \text{if} \sigma_1\leq\mu\leq\sigma_2; \\ -\frac{B_2}{g_3(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m+\frac{\mu B_1^2}{(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m} \\ \qquad -\frac{B_1^2}{g_3(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m \text{if} \mu\geq\sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{g_2^2(n+1)^2}{g_3(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\left\{\frac{(B_2-B_1)+{B_1}^2}{{B_1}^2}\right\},$$

$$\sigma_2 = \frac{g_2^2(n+1)^2}{g_3(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m}\left\{\frac{(B_2+B_1)+{B_1}^2}{{B_1}^2}\right\}.$$

The result is sharp.

Since

$$(\Omega^\gamma D^{n,m}_{\lambda_1,\lambda_2,\ell,d}f)(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\gamma)}{\Gamma(n+1-\gamma)}\left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^m a_n z^n,$$

we have

$$g_2:=\frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)}=\frac{2}{2-\gamma},$$

and

$$g_2:=\frac{\Gamma(4)\Gamma(3-\gamma)}{\Gamma(4-\gamma)}=\frac{6}{(2-\gamma)(3-\gamma)}.$$

Theorem 2 is reduced to Theorem 3 for g_2 and g_3 given by above equalities.

Theorem 3

Let g(z) = z +$\sum_{n=2}^{\infty} g_n z^n$, $(g_n > 0)$ and let the function $\emptyset(z)$ be given by $\emptyset(z) = 1 + \sum_{n=2}^{\infty} B_n z^n$. If the operator given by (4) belongs to $\mathcal{M}^{n,m,g}_{\lambda_1,\lambda_2,\ell,d}(\emptyset)$, then

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{B_2(2-\gamma)(3-\gamma)}{6(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m - \frac{\mu(2-\gamma)_3 B_1^2}{4(3-\gamma)(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m} + \frac{B_1^2(2-\gamma)(3-\gamma)}{6(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m & if \mu \le \sigma_1; \\ \frac{B_1(2-\gamma)(3-\gamma)}{6(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m & if \sigma_1 \le \mu \le \sigma_2; \\ -\frac{B_2(2-\gamma)(3-\gamma)}{6(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m + \frac{\mu(2-\gamma)B_1^2}{4(n+1)^2}\left(\frac{\ell+\ell\lambda_2+d}{\ell+\ell(\lambda_1+\lambda_2)+d}\right)^{2m} - \frac{B_1^2(2-\gamma)(3-\gamma)}{6(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m & if \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{2(3-\gamma)(n+1)^2}{3(2-\gamma)(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m \left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m} \left\{\frac{(B_2-B_1)+{B_1}^2}{{B_1}^2}\right\},$$

$$\sigma_2 = \frac{2(3-\gamma)(n+1)^2}{3(2-\gamma)(n+2)(n+1)}\left(\frac{\ell+2\ell\lambda_2+d}{\ell+2\ell(\lambda_1+\lambda_2)+d}\right)^m \left(\frac{\ell+\ell(\lambda_1+\lambda_2)+d}{\ell+\ell\lambda_2+d}\right)^{2m} \left\{\frac{(B_2+B_1)+{B_1}^2}{{B_1}^2}\right\}.$$

The result is sharp.

Remark 2

When $\ell=1, \lambda_2=d=0$, the above Theorem 3 reduces to a recent result of Al-Shaqsi and Darus [(2008) Theorem 3.3, P. 440]. When $m=n=\lambda_2=\lambda_1=d=0, \ell=1, B_1=\frac{8}{\pi^2}, B_2=\frac{16}{3\pi^2}$, the above Theorem 3 reduces to a recent result of Srivastava and Mishra [(2000), Theorem 8, P.64] for a class of functions for which $\Omega^{\gamma} f(z)$ is a parabolic starlike functions see [Goodman (1991) and RØnning (1993)] .

Conclusions

This paper determined the Fekete-Szegö inequalities for a normalised analytic function $f(z)$ defined on the open unit disc for which $\frac{z(D^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z))'}{D^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)}$ lies in a region starlike with respect to 1 and that it is symmetric with respect to the real axis by using the operator $D^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)$. As a special case of this result, Fekete-Szegö inequality for a class of functions defined by fractional derivatives has been obtained too.

References

- Al-Abbadi, M. H and Darus, M, (2011), The Fekete-Szegö theorem for a certain class of analytic functions. Sains Malaysiana. 40(4): 385-389.

- Al-Shaqsi, K and Darus, M, (2008), On Fekete-Szegö problems for certain subclass of analytic functions. Appl. Math. Sci. 2(9): 431-441.

- Darus, M, (2002), The Fekete-Szegö problems for close-to-convex functions of the class $K_{sh}\ (\alpha\ ,\beta\ )$. Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis 18(1): 13-18.

- Ma, W and Minda, D, (1994), A unified treatment of some speial classes of univalent functions, in: Proceeding of the conference on complex analysis, Z. Li. F.Ren, L.Yang, and S. Zhang (Eds.), Int. Press , 157-169.

- Mohammed A. and Darus M, (2010), On Fekete-Szegö problem for certain subclass of analytic functions. Int. J. Open Problems Compt. Math. 3(4): 510-520.

- Oshah, A and Darus, M, (2014), Differential sandwich theorems with a new generalized derivative operator. Advances in Mathematics: Scientific Journal, 3(2), 117-124.

- RØnning, F, (1993), Uniformly convex function and a corresponding class of starlike functions, Proc. Amer. Math. Soc., 118, 189-196.

- Ravichandran, V, Gangadharan, A. and Darus, M, (2004), On Fekete-Szegö inequality for certain class of Bazilevic functions. Far East Journal of Mathematical Sciences 15(2): 171-180.

- Ramadan, S.F and Darus, M, (2011), On the Fekete- Szegö inequality for a class of analytic functions defined by using generalized differential operator, Acta Universitatis Apulensis, 26, 167-178.

- Srivastava, H. M. and Mishra, A. K, (2000), Applications of fractional calculus to parabolic starlike and uniformly convex functions, Comput. Math. Appl., 39,57-69.

- Srivastava H. M., Mishra A. K. and Das M. K. (2001), The Fekete-Szegö problem for a subclass of close-to-convex functions. Complex Variables, Theory and Application 44: 145-163.

- Srivastava, H. M. and Owa, S, (1989), Univalent functions, fractional calculus and their applications., John Wiley and Sons, New Jersey.