ANALYTICAL SOLUTION FOR DYNAMIC ANALYSIS OF ASYMMETRIC LAMINATED COMPOSITE SHEAR-DEFORMABLE BEAMS

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Abstract

The exact analytical solutions towards investigating the dynamic analysis of extensional-flexural coupled vibration responses for asymmetric composite laminated rectangular beams under various harmonic axial and bending forces are presented.

Three governing coupled differential equations and related boundary conditions were derived from the variational form of Hamilton's principle. The formulations are based on the first order shear-deformable beam theory, account for the effects of rotary inertia, Poisson's ratio, and structural bending-extensional coupling coming from material anisotropy. The resulting coupled equations for asymmetric composite beams were exactly solved and closed-form solutions for extensional-flexural coupled response were obtained for different boundary conditions. Numerical examples were performed for antisymmetric cross-ply and angle-ply laminated composite beams in order to investigate the effects of transverse shear deformation, fibre orientation angle on coupled natural frequencies, quasi-static, and steady state dynamic responses.

Results for dynamic bending and axial displacements are discussed in detail and the validity and accuracy of the present solutions were verified against published exact and finite element solutions.

Keywords: Analytical solution; extensional-flexural coupled response; antisymmetric laminated beams.

Introduction and Objective

Structural members made of composite laminates are increasingly being used in different engineering applications due to their high strength-to-weight and stiffness-to-weight ratios. Multi-layered composite beams are widely used in aerospace, mechanical and civil engineering. Due to their excellent features, composite laminated beams are of the most important structural members used in aircraft wings and fuselage structures, helicopter blades, vehicle axles, propellant and turbine blades, ship and marine structural frames.

In these applications, composite laminated beams are frequently subjected to cyclic dynamic loading (e.g., harmonic excitations). Sources of such forces include aerodynamic effects, hydro-dynamic wave motion and wind loading.

In addition, harmonic forces may arise from unbalanced rotating machinery and propellants, and reciprocating machines. In such applications, composite laminated beams under harmonic forces cause an undesirable vibration and they are prone to fatigue failures.

Fatigue failures are increasingly becoming important to the design of the composite structural members. Under harmonic forces, the transient component of dynamic response is more effective at the beginning of the excitation. Because it has a tendency to dampen out quickly, it is of no importance in evaluating the fatigue life of the composite laminated beam. On the other hand, the steady state dynamic response lasts for a long time hence, it is of particular importance to fatigue life and that is the reason for tackling it within the present study.

Thus, the goal of this study is to develop an efficient solution, which captures and isolates the steady state response. The present analytical closed form solution can also capture the quasi-static response and predict the eigen-frequencies and eigen-modes of the composite antisymmetric laminated beam.

Although the dynamic analysis of the composite laminated antisymmetric beams, which is based on different beam theories, has been the subject of significant research studies during the past few years, most of these studies were restricted to free vibrations of composite antisymmetric laminated beams.

Numerous studies developed and investigated the analytical exact solutions and finite element techniques for free vibration response of composite symmetric and antisymmetric laminated beams. Among the mare Khdeir and Reddy (1994) who developed an exact solution, which is based on higher-order shear deformation theory to study the free vibration behaviour of cross-ply rectangular beams with arbitrary boundary conditions. Banerjee (1998) investigated the free vibration of axially composite laminated Timoshenko beams by using dynamic stiffness matrix method. His exact dynamic stiffness matrix formulation exhibited the coupling between bending and torsion and captured the effects of axial force, shear deformation, and rotatory inertia.

The differential quadrature method is used to obtain the numerical solution of the governing differential equations for symmetrically and antisymmetrically composite beams with rectangular cross-section and for various boundary conditions. Based on the first order shear deformation theory, Chakraborty et al. (2002) used the finite element to analyse the free vibration and wave propagation in composite laminated beams having symmetric and asymmetric ply stacking. Tahani (2007) presented a displacement-based layer wise beam theory and applied it to cross-ply antisymmetric

($0^o/90^o$) and ($0^o/90^o/0^o$) laminated beams subjected to sinusoidal load. Jun et.al. (2008, 2009) developed the exact dynamic stiffness matrix method of free vibration analyses of arbitrary laminated composite beams based on first order shear deformation, trigonometric shear deformation, and higher-order shear deformation beam theories. The effects of shear deformation, rotary inertia, Poisson's ratio, axial force and extensional-bending coupling deformations are considered in their mathematical formulations.

Hjaji et.al. (2016) developed a super-convergent one-dimensional finite beam element with two-nodes for the steady state dynamic flexural response of symmetric laminated composite beams under bending harmonic forces. The new beam element based on the exact shape functions, which satisfy the dynamic coupled governing filed equations, is applicable to symmetric laminated composite beams and accounts for the effects of shear deformation, rotary inertia, and Poison's ratio.

Hjaji et.al. (2017) investigated the analytical closed-from solutions for the flexural dynamic analysis of symmetric laminated composite beams subjected to transverse harmonic forces. Based on the first-order shear deformation theory in which the influences of shear deformation, rotary inertia, Poisson's ratio, and fibre orientation are incorporated in their formulations.

Recently, Horta et.al (2022) investigated the free vibration analysis of laminated composite beams using the finite element method, in which the two-noded Timoshenko beam element model formulated via strain gradient.

Based on finite element method with dynamic finite element techniques, Kashani and Hashemi (2022) presented the free-coupled bending-torsion vibration analysis of prestressed composite laminated beams subjected to static axial force and end moment.

While most of the previous studies focused on free vibration analysis of composite laminated beams, the dynamic analysis of composite laminated beams under dynamic forces was accounted for in a few studies. To the best of the authors' knowledge, no study reported analytical closed-form solutions for the dynamic analysis of composite antisymmetric laminated Timoshenko beams under harmonic forces. Thus, the present study is to formulate the exact closed-form solutions for antisymmetric laminated beams of rectangular cross-sections subjected to harmonic axial and bending forces. The coupled dynamic governing equations and related boundary conditions for the composite antisymmetric laminated beams will be obtained by using Hamilton's variational principle. The effects of shear deformation, rotary inertia, Poisson's ratio and fibre orientation on natural frequencies, quasi-static and steady state dynamic responses are to be investigated too. Several computer programs coded in Maple software by the researchers will be used to compute the numerical results. The present exact solutions are suitable and efficient in analysing the forced bending vibration of composite antisymmetric laminated beams subjected to harmonic axial and bending forces.

Mathematical Formulation

The mathematical model of the fully coupled antisymmetric composite Timoshenko beam in this study is based on the following assumptions:

1. The material of composite beam is linearly elastic.
2. Each lamina is thin and perfectly bonded.
3. Displacements, strains, and rotations are assumed small.
4. The beam cross-section is rigid; in-plane or out-of-plane warping deformations are taken into account.
5. Plane sections normal to the beam axis remain plane before deformation, but not necessarily remain normal to the beam axis after deformation.
6. Only the steady state dynamic response is sought.
7. Damping effect is neglected.

1. Kinematic Relations

A prismatic multi-layered composite beam with length L, thickness h , and width b, as shown in Figure (1), was considered. The right-handed Cartesian coordinate system (X, Y, Z) was defined on the mid-plane of the composite beam, the X axis was coincident with the beam axis, and Y , and Z were coincident with the principal axes of the cross-section. Since the cross-section of the composite beam have two axes of symmetry (i.e., $Y$and Z), the coupling between bending and torsion responses due to the section non-symmetry is neglected i.e. the present study is restricted to flexural behaviour in the $X - Z$ plane. Thus, the displacement fields for a general point $p(x, z)$ of height z from the centroidal axis of composite beam based on the first order shear deformation theory are assumed to take the form:

$$\boldsymbol{u_p(x, z, t) = u(x, t) + z\,\phi_x,\ v_p(x, z, t) = 0}, \textbf{ and } \boldsymbol{w_p(x, z, t) = w(x, t)} \qquad \textbf{(1-3)}$$

in which $u(x, t)$ and $w(x, t)$ are the axial and transverse displacements of a point on the mid-plane in the X and Zdirections, $u_p(x, z, t)$ and $w_p(x, z, t)$ are the axial and transverse displacement, respectively, $v_p(x, z, t)$ is the lateral displacement, and $\phi_x(x, t)$ is the rotation of the normal to the mid-plane about the Y axis, where x and t are spanwise coordinate and time, respectively.

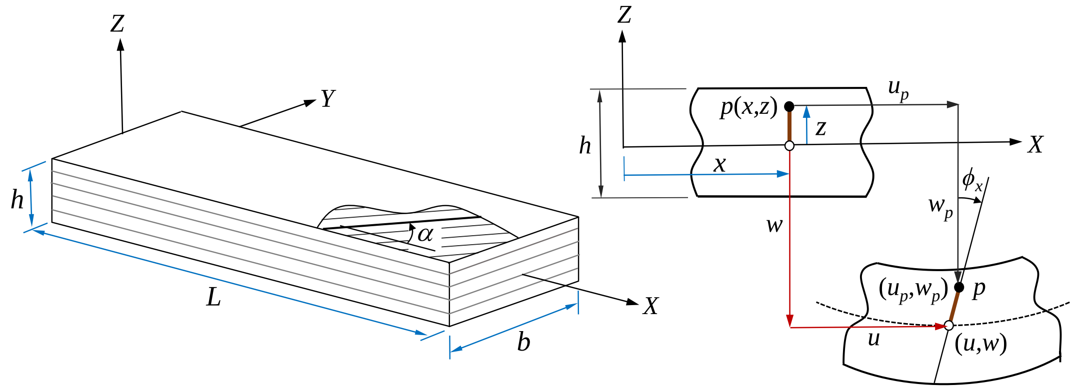

Figure (1): Laminated Composite Beam with Undeformed and Deformed Configurations.

2. Strain-Displacement Relations

The strain relations of the composite beam associated with the small-displacement theory of elasticity are given as:

$$\boldsymbol{\varepsilon_x \approx \partial u_p/\partial x = \varepsilon_{xo} + z k_x, \qquad \gamma_{xz} \approx \partial w_p/\partial x + \phi_x} \tag{4}$$

where $\varepsilon_{xo} = \partial u/\partial x = u'$ is the mid-plane axial strain, $k_x = \partial\phi_x/\partial x = \phi_x'$ is the bending curvature, and the primes denote the differentiation with respect to$x$.

3. Constitutive Equations for Anti-symmetric Laminated Beam

The laminated beam constitutive equations based on the first order shear deformation theory can be obtained by using the classical lamination theory to give:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}_{6\times1} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}_{6\times6} \begin{Bmatrix} \varepsilon_{xo} \\ \varepsilon_{yo} \\ \gamma_{xy} \\ k_x \\ k_y \\ k_{xy} \end{Bmatrix}_{6\times1} \tag{5}$$

where N_x, N_y and N_{xy} are the in-plane forces, M_x, N_y and M_{xy} are the bending and twisting moments, $\varepsilon_{xo}, \varepsilon_{yo}$ and γ_{xy} are the mid-plane strains, k_x, k_y and k_{xy} are the bending and twisting curvatures, respectively, A_{ij}, B_{ij} and D_{ij} denote the extensional, bending-extensional coupling and bending stiffness, respectively, and are expressed as functions of laminate ply orientation and material properties:

$$\boldsymbol{A_{ij}, B_{ij}, D_{ij} = \int_{-h/2}^{h/2} [\bar{Q}_{ij}](1, z, z^2)\, dz \;, \quad (\text{for } i,j = 1,2,6)} \tag{6}$$

where $\overline{Q}_{ij}$ are the transformed reduced stiffnesses and are given by the following expressions Jun et.al, (2008):

$$\bar{Q}_{11} = Q_{11}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2 + Q_{22}s^4$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(s^4 + c^4)$$

$$\bar{Q}_{22} = Q_{11}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2 + Q_{22}c^4$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\, s\, c^3 + (Q_{12} - Q_{22} + 2Q_{66})s^3c$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})s^3c + (Q_{12} - Q_{22} + 2Q_{66})\, s\, c^3$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4)$$

where β is the angle between the fibre direction and longitudinal axis of the composite beam Figure (1), $s = sin\beta$, c=$cos\beta$, and $Q_{11}, Q_{12}, Q_{22}\ and\ Q_{66}$ are the stiffness constants and are given in terms of engineering elastic constants by:

$$Q_{11} = E_{11}/(1 - \upsilon_{12}\upsilon_{21})\,, Q_{12} = \upsilon_{21}\, E_{11}/(1 - \upsilon_{12}\upsilon_{21}) = \upsilon_{12}\, E_{22}/(1 - \upsilon_{12}\upsilon_{21}),$$

$$Q_{22} = E_{22}/(1 - \upsilon_{12}\upsilon_{21})\,, Q_{66} = G_{12}.$$

where the constants E_{11}, and E_{22} are Young moduli, G_{12}, G_{13}, and G_{23} are shear moduli, and $\upsilon_{12}, \upsilon_{21}$ are Poison ratios measured in the principal axes of the layer.

The present formulation is based on first order shear deformation theory in which the effect of transverse shear deformation due to bending is incorporated, then, the transverse shear force per unit length Q_{xz} is given by Vo and Thai (2012):

$$\boldsymbol{Q_{xz} = A_{55}\gamma_{xz} = A_{55}(\partial w/\partial x + \phi_x) = A_{55}(w' + \phi_x)} \qquad \textbf{(7)}$$

in which $A_{55} = k\int_{-h/2}^{h/2} \bar{Q}_{55}\, dz$, where $\bar{Q}_{55} = G_{13}c^2 + G_{23}s^2$, k is the correlation shear factor and is taken as 5/6 to account for the parabolic variation of the transverse shear stresses.

The composite laminated beam is subjected to axial and bending dynamic forces. Then, the lateral in-plane forces and moments in$Y$direction are negligible and set to zero, i.e., $N_y = N_{xy} = M_y = M_{xy} = 0$. In order to account for Poisson's ratio, the mid-plane strains $\varepsilon_{yyo}, \gamma_{xy}$ and curvatures k_{yy}, k_{xy} are assumed to be non-zero. Thus, equation (5) can be rewritten as:

$$\begin{Bmatrix} \boldsymbol{N_x} \\ \boldsymbol{M_x} \end{Bmatrix}_{2\times1} = \begin{bmatrix} \boldsymbol{\bar{A}_{11}} & \boldsymbol{\bar{B}_{11}} \\ \boldsymbol{\bar{B}_{11}} & \boldsymbol{\bar{D}_{11}} \end{bmatrix}_{2\times2} \begin{Bmatrix} \boldsymbol{\varepsilon_{xxo}} \\ \boldsymbol{k_x} \end{Bmatrix}_{2\times1} = \begin{bmatrix} \boldsymbol{\bar{A}_{11}} & \boldsymbol{\bar{B}_{11}} \\ \boldsymbol{\bar{B}_{11}} & \boldsymbol{\bar{D}_{11}} \end{bmatrix}_{2\times2} \begin{Bmatrix} \boldsymbol{u'} \\ \boldsymbol{\phi'_x} \end{Bmatrix}_{2\times1} \qquad \textbf{(8)}$$

where:

$$\begin{bmatrix} \bar{A}_{11} & \bar{B}_{11} \\ \bar{B}_{11} & \bar{D}_{11} \end{bmatrix}_{2\times 2} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix}_{2\times 2} -$$

$$\begin{bmatrix} A_{12} & A_{16}B_{12} & B_{16} \\ B_{12} & B_{16}D_{12} & D_{16} \end{bmatrix} \begin{bmatrix} A_{22} & A_{26} & B_{22} & B_{26} \\ A_{26} & A_{66} & B_{26} & B_{66} \\ B_{22} & B_{26} & D_{22} & D_{26} \\ B_{26} & B_{66} & D_{26} & D_{66} \end{bmatrix}^{-1} \begin{bmatrix} A_{12} & B_{12} \\ A_{16} & B_{16} \\ B_{12} & D_{12} \\ B_{16} & D_{16} \end{bmatrix}.$$

If the Poisson ratio effect is ignored, the coefficients $(\bar{A}_{11}, \bar{B}_{11}, \bar{D}_{11})$ in equation (8) are then replaced by the laminate stiffness coefficients (A_{11}, B_{11}, D_{11}), respectively.

Energy Expressions

The total kinetic energy $T$of the laminated composite beam is given by:

$$\boldsymbol{T = \frac{1}{2}\int_0^L \int_{-h/2}^{h/2} \rho \left[\dot{u}_p^2 + \dot{v}_p^2 + \dot{w}_p^2\right] bdzdx = \frac{1}{2}\int_0^L \left[I_1\dot{u}^2 + I_2\dot{w}^2 + 2I_2\dot{u}\dot{\phi}_x + I_3\dot{\phi}_x^2\right] bdx} \quad \textbf{(9)}$$

in which the dot denotes the derivative with respect to time, and the densities I_1, I_2 and I_3 of the composite beam are introduced by:

$$I_1, I_2, I_3 = \int_{-h/2}^{h/2} \rho\,[1, z, z^2]\,dz$$

$$= \sum_{k=1}^{m} \rho_n[(z_k - z_{k-1}), (z_k^2 - z_{k-1}^2)/2\,, (z_k^3 - z_{k-1}^3)/3]$$

where ρ_n(for $n = 1{,}2{,}3$) are the mass densities of the k^{th} layers.

The total strain energy U_s of the laminated composite beam are given by:

$$U_s = \frac{1}{2}\int_0^L [N_x\varepsilon_{xxo} + M_x k_x + Q_{xz}\gamma_{xz}]\,bdx$$

$$= \frac{1}{2}\int_0^L [N_x u' + M_x\phi_x' + Q_{xz}(w' + \phi_x)]\,bdx$$

From Equations (6) and (8), by substituting into above equation, yields:

$$\boldsymbol{U_s = \frac{1}{2}\int_0^L [\bar{A}_{11}u'^2 + 2\bar{B}_{11}u'\phi_x' + \bar{D}_{11}\phi_x'^2 + A_{55}(w'^2 + 2w'\phi_x + \phi_x^2)]\,bdx} \quad \textbf{(10)}$$

The work done V by the applied harmonic axial and bending forces can be written as:

$$\boldsymbol{V = -\int_0^L [q_x(x,t)u(x,t) + q_z(x,t)w(x,t) + m_x(x,t)\phi_x(x,t)]\,bdx - [P_x(x_e,t)u(x_e,t)]_0^L - [P_z(x_e,t)w(x_e,t)]_0^L - [M_x(x_e,t)\phi_x(x_e,t)]_0^L} \quad \textbf{(11)}$$

Expressions for Force Functions

The composite laminated beam shown in Figure (2) is assumed to be subjected to (a) distributed harmonic forces and bending moments within the beam and (b) concentrated harmonic forces and bending moments at beam both ends, i.e.,

$$\boldsymbol{q_x(x,t), q_z(x,t), m_x(x,t) = [\bar{q}_x(x),\ \bar{q}_z(x),\ \bar{m}_x(x)]e^{i\Omega t}} \quad \textbf{(12)}$$

$$\boldsymbol{P_x(x_e,t), P_z(x_e,t), M_x(x,t) = [\bar{P}_x(x),\ \bar{P}_z(x),\ \bar{M}_x(x)]e^{i\Omega t}\ , for\ \ x_e = 0, L} \quad \textbf{(13)}$$

where Ω is the circular exciting frequency of the applied forces, $i = \sqrt{-1}$ is the imaginary constant, $q_x(x,t)$ and $q_z(x,t)$ are the distributed axial and transverse harmonic forces, $m_x(x,t)$ is the distributed harmonic bending moment, $P_x(x_e,t)$ and $P_z(x_e,t)$ are the concentrated axial and transverse harmonic forces, $M_x(x_e,t)$ is the concentrated harmonic bending moment, all forces and moments are applied at beam ends $(x_e = 0, L)$.

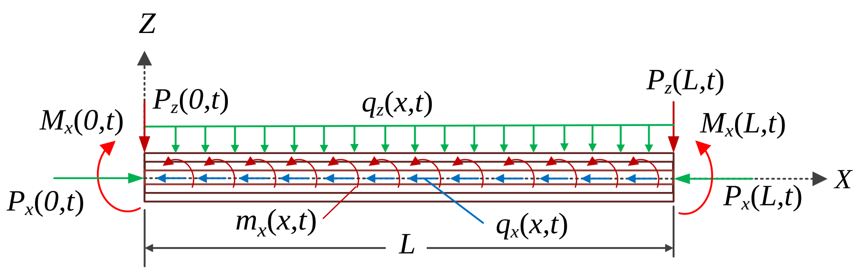

Figure (2): Composite Beam Under General Axial, and Bending Harmonic Forces.

Expressions for Displacement Functions

Under the given applied harmonic forces, the displacement functions corresponding to the steady state component of the dynamic response are assumed to take the forms:

$$\boldsymbol{u(x,t), w(x,t), \phi_x(x,t) = [U(x),\ W(x),\ \Phi_x(x)]e^{i\Omega t}} \quad \textbf{(14)}$$

in which $U(x)$, $W(x)$, and $\Phi_x(x)$ are the amplitudes for axial translation, bending displacement, related bending rotation, respectively. Since the present formulation is intended to capture only the steady state dynamic response of the structural composite beam, the displacement fields postulated in (14) neglect the transient component of the dynamic response.

Variational Principle

The dynamic differential coupled equations for composite antisymmetric laminated beam subjected to harmonic forces can be derived using Hamilton's principle, which can be written as:

$$\int_{t_1}^{t_2} \boldsymbol{\delta(T-\Pi)\,dt = 0\,,\ for\ \delta u(x,t) = \delta w(x,t) = \delta\phi_x(x,t) = 0\ at\ t = t_1\ and\ t_2} \quad \textbf{(15)}$$

where t_1 and t_2 are two arbitrary time variables and δ denotes the first variation.

From equations (12)-(14) and by substituting into energy expressions in (9)-(11), the resulting equations into equation (15), performing integration by parts, the governing equations of motion are obtained in matrix form as:

$$\begin{bmatrix} (I_1\Omega^2 + \bar{A}_{11}\mathcal{D}^2) & 0 & (I_2\Omega^2 + \bar{B}_{11}\mathcal{D}^2) \\ 0 & -(I_1\Omega^2 + A_{55}\mathcal{D}^2) & -A_{55}\mathcal{D} \\ (I_2\Omega^2 + \bar{B}_{11}\mathcal{D}^2) & -A_{55}\mathcal{D} & (I_3\Omega^2 - A_{55} + \bar{D}_{11}\mathcal{D}^2) \end{bmatrix}_{3\times3} \begin{Bmatrix} U(x) \\ W(x) \\ \Phi_x(x) \end{Bmatrix}_{3\times1} = \begin{Bmatrix} -\bar{q}_x(x) \\ \bar{q}_z(x) \\ \bar{m}_x(x) \end{Bmatrix}_{3\times1} \quad \textbf{(16)}$$

The related boundary conditions arising from the variational principle are:

$$[b\bar{A}_{11}U'(x) + b\bar{B}_{11}\Phi_x'(x) - \bar{P}_x(x)]_0^L \delta U(x)|_0^L = 0 \quad \textbf{(17)}$$

$$[bA_{55}(W'(x) + \Phi_x(x)) - \bar{P}_z(x)]_0^L \delta W(x)|_0^L = 0 \quad \textbf{(18)}$$

$$[b\bar{B}_{11}U'(x) + b\bar{D}_{11}\Phi_x'(x) - \bar{M}_x(x)]_0^L \delta\Phi_x(x)|_0^L = 0 \quad \textbf{(19)}$$

where $\mathcal{D}$ is the differential operator, i.e., $\mathcal{D} \equiv d/dx\,, \mathcal{D}^2 \equiv d^2/dx^2$. Equations in (16) govern the coupled extensional-flexural dynamic response of composite antisymmetric laminated beam under harmonic forces. The present study is focused on the exact closed-form solutions for the steady state dynamic response governed by these coupled equations.

It is noted that, the above extensional-flexural coupled equations in (16) with related boundary conditions (17-19) are similar to those derived by Jun et.al. (2008) for free vibration of laminated composite beams when the axial compressive force effect is omitted. The present treatment differs from that in Jun et al. in two respects:

(1) while Jun et.al. (2008) investigated the free vibration analysis of laminated composite beams, the present solution provides the complete steady state dynamic response under general harmonic bending forces with a given exciting frequency.

(2) in the present study, the closed form solutions of steady state dynamic responses are derived in exact expressions, while in Jun et.al. (2008) provided only an analytical solution using dynamic stiffness matrix for determining the natural frequencies and buckling loads for the composite beams.

Exact Solution for Coupled Field Equations

1. Homogeneous Solution

The homogeneous solution of the extensional-flexural coupled Equations in (16) is obtained by setting the right-hand side of the equations to zero, i.e. $\bar{q}_x(x) = \bar{q}_z(x) = \bar{m}_x(x) = 0$. The homogeneous solution of the displacement functions is then assumed to take the form:

$$\boldsymbol{\langle \chi_h(x)\rangle_{1\times3} = \langle U_h(x) \quad W_h(x) \quad \Phi_{xh}(x)\rangle_{1\times3} = \langle C\rangle_{1\times3}\, e^{m_i x}\,,\ for\ i = 1,2,3,\dots,6} \qquad \textbf{(20)}$$

where $\langle \chi(x)\rangle_{1\times3} = \langle U_h(x) \quad W_h(x) \quad \Phi_{xh}(x)\rangle_{1\times3}$ is the vector of extensional, flexural displacement and bending rotation functions, and $\langle C\rangle_{1\times3} = \langle c_{1,i} \quad c_{2,i} \quad c_{3,i}\rangle_{1\times3}$ is the vector of unknown constants. From equation (20), by substituting into the equations in (16), for non-trivial solution, the determinant of the bracketed matrix is set to vanish leading to the sixth-order polynomial equation of the form:

$$\boldsymbol{\beta_4 m_i^6 + \beta_3 m_i^4 + \beta_2 m_i^2 + \beta_1 = 0} \qquad \textbf{(21)}$$

where $\beta_1 = \Omega^4 I_1[\Omega^2(I_1 I_3 - I_2^2) - I_1 A_{55}]$,

$\beta_2 = \Omega^2[\Omega^2 I_1(I_1\bar{D}_{11} + I_3\bar{A}_{11} - 2I_2\bar{B}_{11}) + A_{55}(I_1 I_3\Omega^2 - I_2^2\Omega^2 - I_2\bar{A}_{11}]$,

$\beta_3 = \Omega^2[I_1\bar{D}_{11}(A_{55} + \bar{A}_{11}) + I_3\bar{A}_{11}A_{55} - I_1\bar{B}_{11}^2 - 2I_2\bar{B}_{11}A_{55})]$, and

$\beta_4 = A_{55}(\bar{A}_{11}\bar{D}_{11} - \bar{B}_{11}^2)$

The characteristic equation (21) has six distinct roots m_i $(for\ i = 1,2,3,\dots,6)$. For each root m_i, there corresponds a set of constants $\langle C\rangle_{i,1\times3} = \langle c_{1,i} \quad c_{2,i} \quad c_{3,i}\rangle_{i,1\times3}$. By back-substitution into the homogeneous coupled system of equations in (20), one can relate constants $c_{1,i}$ and $c_{2,i}$ to constants $c_{3,i}$ through $c_{1,i} = G_{1,i}c_{3,i}$ and $c_{2,i} = G_{2,i}c_{3,i}$, (for $i = 1,2,3,\dots,6$), respectively, where $G_{1,i} = -(\bar{B}_{11}m_i^2 + I_2\Omega^2)/(\bar{A}_{11}m_i^2 + I_1\Omega^2)$, and $G_{2,i} = -A_{55}m_i/(A_{55}m_i^2 + I_1\Omega^2)$.

The homogeneous solutions for extensional displacement $U_h(x)$, flexural displacement $W_h(x)$ and related bending rotation $\Phi_{xh}(x)$ are obtained as:

$$\boldsymbol{\{\chi_h(x)\}_{3\times1} = [\bar{G}]_{3\times6}[E(x)]_{6\times6}\{\bar{C}\}_{6\times1}} \qquad \textbf{(22)}$$

in which $[\bar{G}]_{3\times 6} = \begin{bmatrix} \begin{Bmatrix} G_{1,1} \\ G_{2,1} \\ 1 \end{Bmatrix} & \begin{Bmatrix} G_{1,2} \\ G_{2,2} \\ 1 \end{Bmatrix} & \begin{matrix} \cdots\cdot \\ \cdots\cdot \\ \cdots\cdot \end{matrix} & \begin{Bmatrix} G_{1,6} \\ G_{2,6} \\ 1 \end{Bmatrix} \end{bmatrix}_{3\times 6}$, $[E(x)]_{6\times 6}$ is a diagonal matrix consisting of the exponential functions $e^{m_i x}$ (for $i = 1,2,3,\dots.,6$), the vector of unknown integration constants $\langle \bar{C} \rangle_{1\times 6} = \langle c_{3,1} \quad c_{3,2} \quad \cdots\cdots \quad c_{3,6} \rangle_{1\times 6}$ is to be determined from the problem boundary conditions.

2. Particular Solution for Uniform Member Harmonic Forces

For a composite antisymmetric laminated beam under uniform distributed axial and bending harmonic forces $\left(\bar{q}_x(x), \bar{q}_z(x), \bar{m}_x(x)\right)e^{i\Omega t} = (\bar{q}_x, \bar{q}_z, \bar{m}_x)e^{i\Omega t}$, the corresponding particular solution $\langle \chi_p \rangle_{1\times 3} = \langle U_p \quad W_p \quad \Phi_{xp} \rangle_{1\times 3}$ of the coupled equations in (16) is assumed to take the form:

$$\langle \boldsymbol{\chi_p} \rangle_{\mathbf{1}\times\mathbf{3}} = \langle \boldsymbol{U_p} \quad \boldsymbol{W_p} \quad \boldsymbol{\Phi_{xp}} \rangle_{\mathbf{1}\times\mathbf{3}} = \langle \boldsymbol{A_1} + \boldsymbol{B_1 x} \quad \boldsymbol{A_2} + \boldsymbol{B_2 x} \quad \boldsymbol{A_3} + \boldsymbol{B_3 x} \rangle_{\mathbf{1}\times\mathbf{3}} \qquad \textbf{(23)}$$

From expressions in equation (24), by substituting into equation (16), leads to:

$$\langle \boldsymbol{\chi_p} \rangle_{\mathbf{1}\times\mathbf{3}} = \langle \frac{-\bar{\boldsymbol{q}}_x}{\boldsymbol{I_1}\boldsymbol{\Omega}^2} + \frac{\boldsymbol{I_2}}{\boldsymbol{I_1}}\left(\frac{\boldsymbol{I_2}\bar{\boldsymbol{q}}_x + \boldsymbol{I_1}\bar{\boldsymbol{m}}_x}{\boldsymbol{\Omega}^2[\boldsymbol{I_2^2} - \boldsymbol{I_1 I_3}] - \boldsymbol{I_1 A_{55}}}\right) \quad \frac{-\bar{\boldsymbol{q}}_z}{\boldsymbol{b I_1}\boldsymbol{\Omega}^2} \quad \left(\frac{\boldsymbol{I_2}\bar{\boldsymbol{q}}_x + \boldsymbol{I_1}\bar{\boldsymbol{m}}_x}{\boldsymbol{\Omega}^2[\boldsymbol{I_1 I_3} - \boldsymbol{I_2^2}] - \boldsymbol{I_1 A_{55}}}\right) \rangle_{\mathbf{1}\times\mathbf{3}} \qquad \textbf{(24)}$$

The complete exact closed-form solution for the system of extensional-flexural coupled equations is then obtained by adding the homogeneous part in equation (22) to particular part in equation (24) as:

$$\{\boldsymbol{\chi}(\boldsymbol{x})\}_{\mathbf{3}\times\mathbf{1}} = \{\boldsymbol{\chi_h}(\boldsymbol{x})\}_{\mathbf{3}\times\mathbf{1}} + \left\{\boldsymbol{\chi_p}\right\}_{\mathbf{3}\times\mathbf{1}} = [\bar{\boldsymbol{G}}]_{3\times 6}[\boldsymbol{E}(\boldsymbol{x})]_{6\times 6}\{\bar{\boldsymbol{C}}\}_{6\times\mathbf{1}} + \left\{\boldsymbol{\chi_p}\right\}_{\mathbf{3}\times\mathbf{1}} \qquad \textbf{(25)}$$

3. Solution for Antisymmetric Laminated Cantilever Beam

A cantilever composite beam subjected to (i) concentrated end harmonic forces; axial compressive force $\bar{P}_x(L)e^{i\Omega t}$, transverse force $\bar{P}_z(L)e^{i\Omega t}$, end bending moment $\bar{M}_x(L)e^{i\Omega t}$, and (ii) distributed harmonic forces; axial force $\bar{q}_x e^{i\Omega t}$, and transverse force $\bar{q}_z e^{i\Omega t}$.

Imposing the following cantilever boundary conditions at both ends, i.e., $x = 0 \; and \; x = L$:

$$\delta U(0) = \delta W(0) = \delta\Phi_x(0) = 0, \; [\bar{A}_{11}U'(L) + \bar{B}_{11}\Phi_x'(L)] = \bar{P}_x(L),$$

$$A_{55}[W'(L) + \Phi_x(L)] = \bar{P}_z(L), \text{ and } [\bar{B}_{11}U'(L) + \bar{D}_{11}\Phi_x'(L)] = \bar{M}_x(L),$$

Substituting the displacement functions in Equation (25) into above boundary conditions, the total closed form solution for cantilever laminated composite beam is then obtained as:

$$\{\boldsymbol{\chi_c}(\boldsymbol{x})\}_{\mathbf{3}\times\mathbf{1}} = [\bar{\boldsymbol{G}}]_{3\times 6}[\boldsymbol{E}(\boldsymbol{x})]_{6\times 6}[\boldsymbol{\Psi_c}]^{-\mathbf{1}}_{6\times 6}\{\boldsymbol{Q_c}\}_{6\times\mathbf{1}} + \left\{\boldsymbol{\chi_p}\right\}_{\mathbf{3}\times\mathbf{1}} \qquad \textbf{(26)}$$

where $\langle Q_c \rangle_{1\times6} = \langle -U_p \quad -W_p \quad -\Phi_{xp} \quad \bar{P}_x(L) \quad \bar{P}_z(L) - A_{55}\Phi_{xp} \quad \bar{M}_x(L)\rangle_{1\times6}$, and $[\boldsymbol{\Psi}_c]^T_{6\times6} =$
$\left[\boldsymbol{G}_{1,i} \quad \boldsymbol{G}_{2,i} \quad \mathbf{1} \quad \left(\bar{\boldsymbol{A}}_{11}\boldsymbol{G}_{1,i} + \bar{\boldsymbol{B}}_{11}\right)\boldsymbol{m}_i e^{\boldsymbol{m}_i \boldsymbol{L}} \quad \boldsymbol{A}_{55}\left(\boldsymbol{m}_i\boldsymbol{G}_{2,i} + \mathbf{1}\right)e^{\boldsymbol{m}_i \boldsymbol{L}} \quad \left(\bar{\boldsymbol{B}}_{11}\boldsymbol{G}_{1,i} + \bar{\boldsymbol{D}}_{11}\right)\boldsymbol{m}_i e^{\boldsymbol{m}_i \boldsymbol{L}}\right]^{\boldsymbol{T}}_{\mathbf{6\times6}}$

4. Solution for Composite Laminated Simply-Supported Beam

A simply supported composite laminated beam subjected to (a) distributed harmonic forces: axial force $\bar{q}_x e^{i\Omega t}$, transverse force $\bar{q}_z e^{i\Omega t}$, bending moments $\bar{m}_x e^{i\Omega t}$, and (2) end harmonic bending moments $\bar{M}_x(x_e)e^{i\Omega t}$ at beam both ends $(x_e = 0 \; and \; L)$ is considered.

For simply supported beam, the boundary conditions at both ends $(i.e., \; x_e = 0 \; and \; L)$ are:

$$\delta U(0) = \delta W(0) = 0, \; [\bar{B}_{11}U'(0) + \bar{D}_{11}\Phi_x'(0)] = \bar{M}_x(0), \; [\bar{A}_{11}U'(L) + \bar{B}_{11}\Phi_x'(L)] = 0,$$

$$\delta W(L) = 0, \text{ and} [\bar{B}_{11}U'(L) + \bar{D}_{11}\Phi_x'(L)] = -\bar{M}_x(L),$$

From equation (25), by substituting into the above boundary conditions, the general closed form steady state solution for simply supported laminated composite beam is obtained as:

$$\{\boldsymbol{\chi}_s(\boldsymbol{x})\}_{\mathbf{3\times1}} = [\bar{\boldsymbol{G}}]_{\mathbf{3\times6}}[\boldsymbol{E}(\boldsymbol{x})]_{\mathbf{6\times6}}[\boldsymbol{\Psi}_s]^{-\mathbf{1}}_{\mathbf{6\times6}}\{\boldsymbol{Q}_s\}_{\mathbf{6\times1}} + \{\boldsymbol{\chi}_{\boldsymbol{p}}\}_{\mathbf{3\times1}} \qquad \textbf{(27)}$$

where $\langle Q_s \rangle_{1\times6} = \langle -U_p \quad -W_p \quad \bar{M}_x(0) \quad 0 \quad -W_p \quad -\bar{M}_x(L)\rangle_{1\times6}$, and $[\Psi_s]^T_{6\times6} = \left[G_{1,i} \quad G_{2,i} \quad m_i\alpha_i \quad \left(\bar{A}_{11}G_{1,i} + \bar{B}_{11}\right)m_i e^{m_i L} \quad G_{2,i}\, e^{m_i L} \quad \alpha_i m_i e^{m_i L}\right]^T_{6\times6}$ in which $\alpha_i = \left(\bar{B}_{11}G_{1,i} + \bar{D}_{11}\right)$.

5. Solution for Clamped-pinned Composite Beam

Consider a clamped-pinned composite beam under distributed harmonic forces; transverse force $\bar{q}_z(x)e^{i\Omega t}$, bending moment $\bar{m}_x(x)\, e^{i\Omega t}$, and concentrated bending moment $\bar{M}_x(x)\, e^{i\Omega t}$ applied at beam right end $(i.e., x = L)$.

Imposing the related boundary conditions at beam end $(x = 0)$: $U(0) = W(0) = \Phi_x(0) = 0$, and at end $(x = L)$: $[\bar{A}_{11}U'(L) + \bar{B}_{11}\Phi_x'(L)] = 0, W(L) = 0$, and $[\bar{B}_{11}U'(L) + \bar{D}_{11}\Phi_x'(L)] = -\bar{M}_x(L)$, the total steady state solution for clamped-pinned composite beam under given harmonic forces is determined by substituting the axial, transverse bending and bending rotation functions in equations (26) into the above boundary conditions, yields:

$$\{\boldsymbol{\chi}_{cs}(\boldsymbol{x})\}_{\mathbf{3\times1}} = [\bar{\boldsymbol{G}}]_{\mathbf{3\times6}}[\boldsymbol{E}(\boldsymbol{x})]_{\mathbf{3\times6}}[\boldsymbol{\Psi}_{cs}]^{-\mathbf{1}}_{\mathbf{6\times6}}\{\boldsymbol{Q}_{cs}\}_{\mathbf{6\times1}} + \{\boldsymbol{\chi}_{\boldsymbol{p}}\}_{\mathbf{3\times1}} \qquad \textbf{(28)}$$

in which $\langle Q_{cs}\rangle_{1\times 6} = \langle -U_p \quad -W_p \quad -\Phi_{xp} \quad 0 \quad -W_p \quad -M_x(L)\rangle_{1\times 6}$, and $[\Psi_{cs}]_{6\times 6}^T = [G_{1,i} \quad G_{2,i} \quad 1 \quad \alpha_i m_i e^{m_i L} \quad G_{2,i} e^{m_i L} \quad \alpha_i m_i e^{m_i L}]_{6\times 6}^T$.

6. Solution for Clamped-clamped Composite Beam

A clamped-clamped composite laminated beam under distributed harmonic forces; axial compressive force $\bar{q}_x e^{i\Omega t}$, transverse force $\bar{q}_z e^{i\Omega t}$ and bending moment $\bar{m}_x e^{i\Omega t}$ is considered. The beam has the following boundary conditions at both ends, i.e.: $\delta U(x) = \delta W(x) = \delta\Phi_x(x) = 0 \; at \; x = 0 \; and \; L$.

By substituting the displacement functions into the above boundary conditions, the total closed form solution for the clamped-clamped laminated composite beam is then found as:

$$\{\boldsymbol{\chi}_{cc}(\boldsymbol{x})\}_{\mathbf{3\times 1}} = [\overline{\boldsymbol{G}}]_{\mathbf{3\times 6}}[\boldsymbol{E}(\boldsymbol{x})]_{\mathbf{3\times 6}}[\boldsymbol{\Psi}_{cc}]_{\mathbf{6\times 6}}^{-\mathbf{1}}\{\boldsymbol{Q}_{cc}\}_{\mathbf{6\times 1}} + \{\boldsymbol{\chi}_p\}_{\mathbf{3\times 1}} \qquad \textbf{(29)}$$

in which $\langle Q_{cc}\rangle_{1\times 6} = \langle -U_p \quad -W_p \quad -\Phi_{xp} \quad -U_p \quad -W_p \quad -\Phi_{xp}\rangle_{1\times 6}$, and $[\Psi_{cc}]_{6\times 6}^T = [G_{1,i} \quad G_{2,i} \quad 1 \quad G_{1,i}\, e^{m_i L} \quad G_{2,i} e^{m_i L} \quad e^{m_i L}]_{6\times 6}^T$.

Numerical Examples

The analytical closed-form solutions developed in the present study are used to provide the steady state dynamic responses of composite asymmetric laminated beams under various harmonic axial and bending forces. The quasi-static response of the composite beams under harmonic axial and bending forces can be approached by using very low exciting frequency $\Omega \approx 0.01\omega_1$ related to the first natural frequency ω_1 of the composite beam. In order to show the validity, accuracy, and applicability of the present analytical solution, several examples are conducted for asymmetric composite beams having various boundary conditions. In these examples, the laminates have the same thickness and are made of the same orthotropic composite material properties. The results obtained from the present analytical closed-form solution are compared with available exact solutions in the literature and established Abaqus finite shell element. In Abaqus model, the shell S4R element has six degrees of freedom at each node (i.e., three translations and three rotations) and captures the transverse shear deformation effects.

Example (1): Asymmetric Laminated Composite beam under harmonic Forces

This example has been utilized by many researchers Tahani, (2007); Hjaji et.al., (2016); Jun and Hongxing, (2009) for the validation purposes. In order to establish the exactness and validity of the present analytical closed-form solution, a graphite-epoxy asymmetric laminated composite beam with span length of 0.381m and rectangular cross-section (width $b = 25.4mm$ and thickness $h = 25.4mm$) is subjected to uniformly distributed harmonic transverse force $q_z(x,t) = 8.0e^{i\Omega t}\, kN/m$ and bending moment $m_x(x,t) = 6.0e^{i\Omega t}\ kNm/m$ as shown in Figure (3). The composite laminated beam having various boundary conditions is considered. All fibre angles arranged to (30°/50°/30°/50°) and the four plies have the same thickness and made of the same orthotropic composite material as: $E_{11} = 144.8GPa$, $E_{22} = 9.65GPa$, $G_{12} = G_{13} = 4.14GPa$, $G_{23} = 3.45GPa$, $\upsilon_{12} = 0.3$, and $\rho = 1389.2\ kg/m^3$.

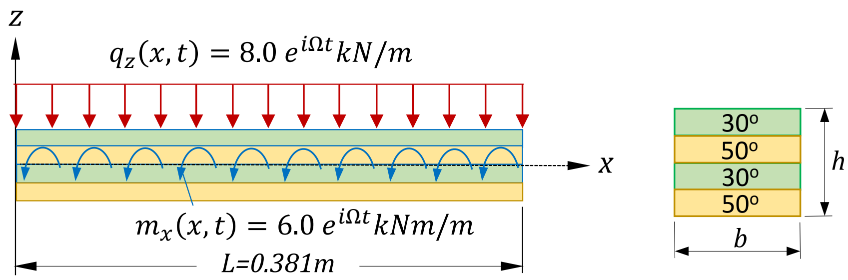

Figure (3): A Composite Laminated Beam Under Harmonic Distributed Bending Forces.

Under uniformly distributed harmonic bending force: $q_z(x,t) = 8.0e^{i\Omega t}kN/m$ and bending moment $m_x(x,t) = 6.0e^{i\Omega t}kNm/m$, the natural frequencies related to the bending response can be extracted from the steady state dynamic analysis when the exciting frequency f is varied from nearly zero to 5000*Hz*. Figures (4a-c) and (4d-f) demonstrate the peak transverse displacement W, axial displacement U, and bending rotation ϕ_x at the midspan $(x = L/2)$ of cantilever and clamped-roller support composite beams as a function of exciting frequency f. Peaks on the diagrams indicate the resonance and the natural frequencies of the given composite beams having cantilevered, and clamped-roller support boundary conditions. Then, the first five natural frequencies extracted at the peaks of Figure (4) are provided in Table (1) for cantilever, and clamped-roller support beams as well as for clamped-clamped boundary conditions.

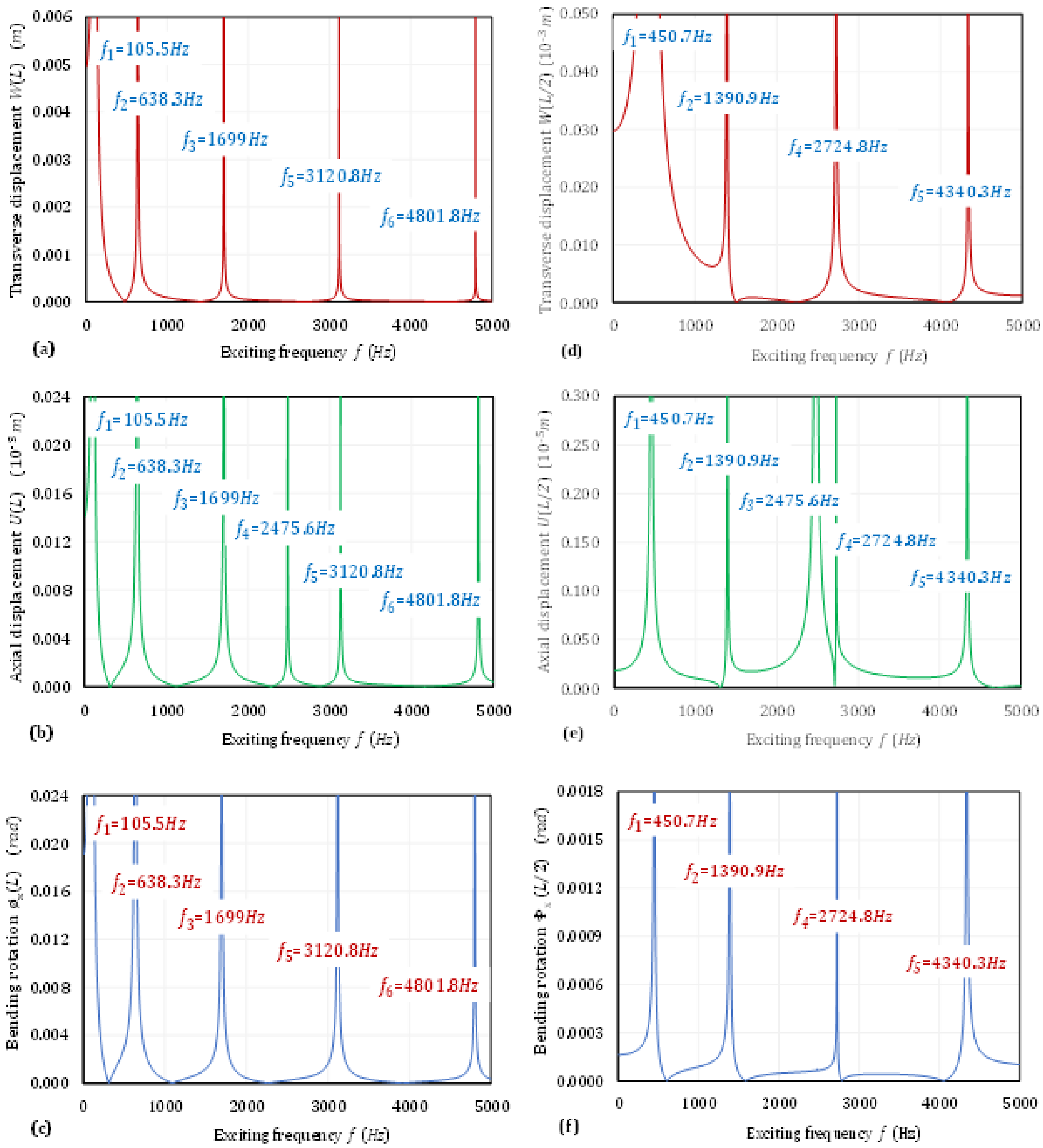

Figure (4): Natural Frequencies of Composite Asymmetric Laminated (30º/50º/30º/50º) Cantilever and Clamped-Roller Support Beams Under Distributed Harmonic Forces.

To illustrate the accuracy of the present closed-form solution, the values of the natural coupled extensional-flexural frequencies obtained from the present formulation in Table (1) are compared with the corresponding results given in Jun et.al. (2008). It is noted that, the present closed-form solution exhibit excellent agreement when compared with those given in Jun et.al. (2008). Accordingly, the present solution is able to capture the eigen-frequencies of the given composite antisymmetric laminated beams with cantilever and clamped-roller support boundary conditions. Additionally, the present solution is capable of obtaining the axial natural frequency (third one) of the composite clamped-roller support beam, while the solution of Jun et.al. (2008) did not capture this frequency.

Table (1): Natural Frequencies for Composite Asymmetric ($30^o/50^o/30^o/50^o$) Laminated Beam.

Boundary condition	Frequency Number	Natural frequencies in (*Hz*)		% Difference =[2-1]/2
		Reference Jun et.al, (2008)	Present Solution	
Cantilever	1	105.4	105.5	0.09%
	2	638.2	638.3	0.02%
	3	1679.0	1679.0	0.00%
	4	2475.5	2475.6*	0.00%
	5	3120.7	3120.8	0.00%
Clamped-roller support	1	451.0	450.7	-0.07%
	2	1391.0	1390.9	-0.01%
	3	-	2475.4*	-
	4	2724.8	2724.8	0.00%
	5	4338.6	4340.3	0.04%
* *Fully Axial natural frequency*				

Example (2): Quasi-static and Dynamic Responses

To validate and confirm the accuracy of present analytical closed-form solution to approach the quasi-static and steady state dynamic responses, the numerical results calculated in this example were compared with those data given in Khdeir and Reddy (1994), Chakraborty et.al. (2002), Vo and Thai (2012), and with Abaqus finite element model.

A composite two-layered asymmetric cross-ply $(0^o/90^o)$ laminated composite beam (with $b = 25.4mm$ and thickness $h = 25.4mm$) which is subjected to the distributed transverse harmonic force $q_z(x,t) = 200e^{i\Omega t}N/m$ was analysed for different values of length to thickness ratio L/h. The quasi-static response of the composite beam under harmonic transverse force was captured using very low exciting frequency $(\text{i.e.}, \Omega \approx 0.01\omega_1)$ related to the first natural frequency ω_1 of the composite beam, while the steady state dynamic response is computed by using an exciting frequency $\Omega = 1.80\omega_1$, where the first natural frequency of the given composite beam was obtained as $\omega_1 = 182.2\,rad/sec$. For the sake of comparison, the asymmetric laminated composite beam, which has clamped-free and simply supported boundary conditions, was considered. The two layers have the same thickness and made of the same orthotropic material properties: $E_{11} = 25.0GPa, E_{22} = 1.0GPa, G_{12} = G_{13} = 0.5E_{22},\ G_{23} = 0.2E_{22},\ \upsilon_{12} = 0.25,\ \rho = 1389.2kg/m^3$.

Quasi-Static Analysis

The transverse displacement function $W(x)$ for quasi-static response of asymmetric cross-ply laminated beam, which is based on the present closed-form solution, was calculated in the non-dimensional form in Vo and Thai (2012) as $\bar{W} = 100bh^3E_{22}W/q_zL^4$, and it was compared with the finite element and exact static solutions given by Khdeir and Reddy (1994), Chakraborty et.al. (2002), and Vo and Thai (2012). The static results of mid-span displacements for different L/h ratios presented in Chakraborty et.al. (2002) and Vo and Thai (2012) were based on finite element formulations, while the corresponding results in Khdeir and Reddy (1994) were based on exact solution.

Table (2) provides the non-dimensional mid-span transverse displacements $\bar{W}(L/2)$ for cantilever and simply supported asymmetric $(0^o/90^o)$ composite beams under distributed transverse forces for different span-to height ratios of (*L/h*)= 5,10,20 and 50. Also the effect of Poison ratio on the static results is presented in Table (2).It is obvious that the static results obtained from the present formulation indicate excellent agreement with results based on other solutions available in the literature.

Table (2): Static Results for non-Dimensional Displacement of Asymmetric Cross-Ply ($0^o/90^o$) Beam Under Distributed Forces with Cantilever and Simply-Supported Boundary Conditions.

Beam Type	Reference		$\bar{W}(x=L/2)$			
			(*L/h*) = 5	**(*L/h*) = 10**	**(*L/h*) = 20**	**(*L/h*) = 50**
Cantilever	Khdeir and Reddy, (1994)		16.436	12.579	-	11.345
	Chakraborty et.al. (2002)		16.496	12.579	-	11.345
	Vo and Thai (2012)		16.461	12.604	11.640	11.370
	Present Solution	Poison ratio included	16.448	12.591	11.626	11.357
		Poison ratio excluded	16.436	12.579	11.615	11.345
Simply supported	Khdeir and Reddy (1994)		5.036	3.750	-	3.339
	Chakraborty et.al. (2002)		5.048	3.751	-	3.353
	Vo and Thai (2012)		5.043	3.757	3.436	3.346
	Present Solution	Poison ratio included	5.040	3.752	3.432	3.342
		Poison ratio excluded	5.036	3.750	3.428	3.339

The quasi-static and steady state dynamic results for the axial and transverse displacements and related bending rotation plotted against the beam coordinate axis

x for span to height ratio $L/h = 20$, in the case of cantilever composite beam, are presented on figures (5a-b) and (5c-d) respectively. It is observed that, the results for quasi-static and dynamic responses obtained from the present closed-form solution demonstrate an excellent agreement with those results based on Abaqus finite element model using 60 beam B31 elements with 366 degrees of freedom.

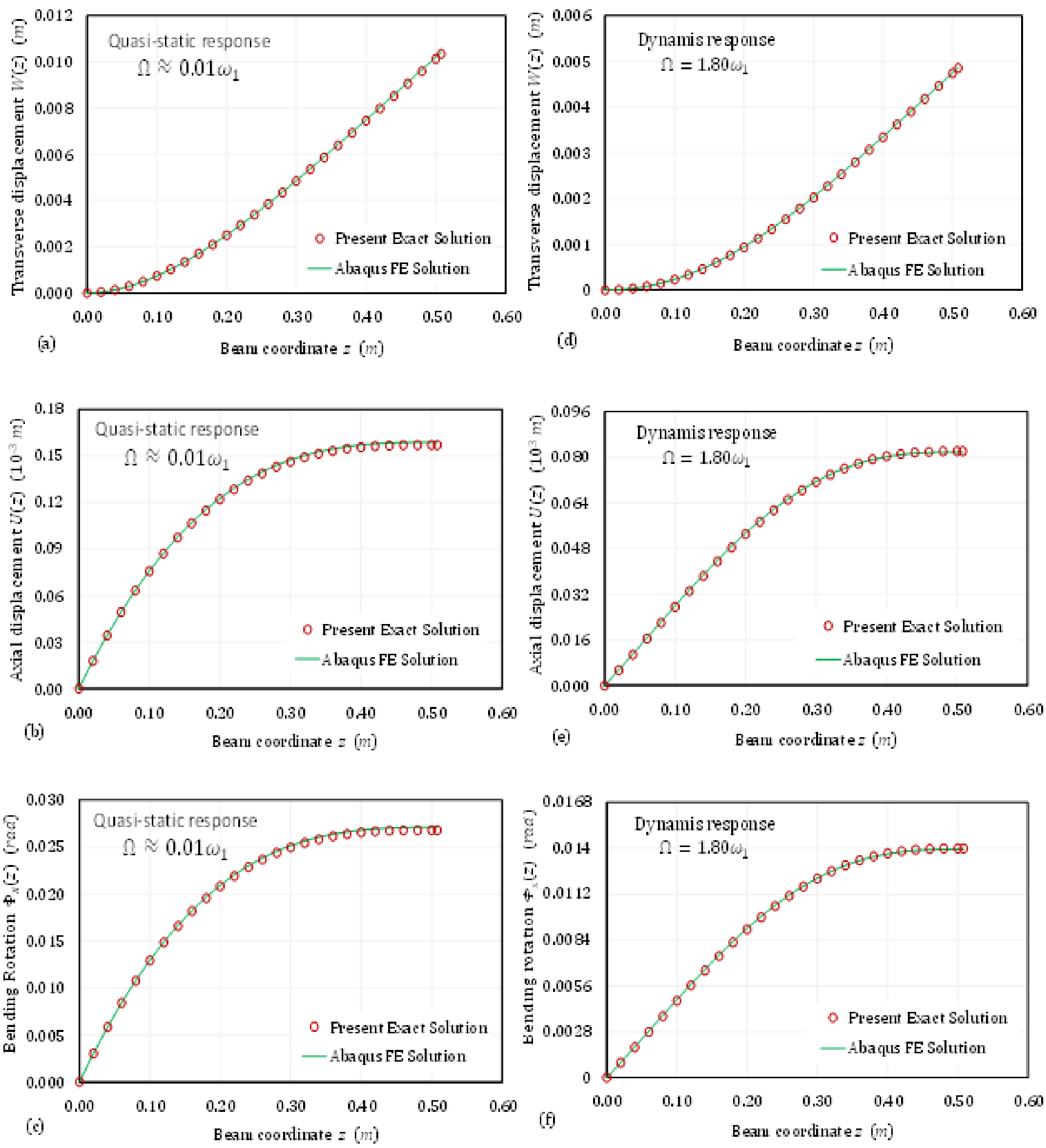

Figure (5): Quasi-Static and Dynamic Responses of Composite Asymmetric (0°/90°) Laminated Cantilever Beam Under Distributed Transverse Harmonic Force.

Example (3): Asymmetric Laminated Beam under Harmonic Forces

Four-layered asymmetric cross-ply $(0^o/90^o/0^o/90^o)$ laminated composite clamped-roller supported beam of 2.40m length were subjected to distributed transverse harmonic forces $q_z(x,t) = 6.0e^{i\Omega t}\ kN/m$ is considered. The four plies have the same thickness and made of the same orthotropic composite material properties: $E_{11} = 144.8\ GPa,\ E_{22} = 9.65\ GPa,\ G_{12} = G_{13} = 4.14\ GPa,\ G_{23} = 3.45\ GPa,\ \upsilon_{12} = 0.30,$ and $\rho = 1550.1\ kg/m^3$. The example is given to:

(i) compute the static response of the composite beam using very low exciting frequency $\Omega \approx 0.01\omega_1$, where the first natural frequency of the given composite beam is $f_1 = 20.39Hz$, and

(ii) determine the steady state dynamic response of the beam under harmonic force at exciting frequency $\Omega = 2.4\omega_1$.

Quasi-static solution

The quasi-static response results for extensional displacement $U(x)$, transverse displacement $W(x)$, and bending rotation $\Phi_x(x)$ are illustrated in Figures (6a,b,c) and (6d,e,f) for composite beams with clamped-roller and clamped-clamped boundary conditions, respectively. The static results are based on the present closed-form solution and Abaqus B31 beam element solution. Results obtained from the present solution provide an excellent agreement with the corresponding results based on Abaqus beam model solution.

Steady state dynamic solution

For the exciting frequency $f = 2.40\ f_1$, (where $f_1 = 20.39Hz$ for clamped-roller beam and $f_1 = 31.19Hz$ for clamped-clamped beam) the steady state dynamic response results for extensional displacement $U(x)$, transverse displacement $W(x)$, and associated bending rotation $\Phi_x(x)$ versus the beam coordinate axis (x) are shown in Figures (7a,b,c) and (7d,e,f) for clamped-roller and clamped-clamped boundary conditions, respectively. Again, the dynamic results obtained from the present closed-form solution are in excellent agreement with the Abaqus B31 beam element results.

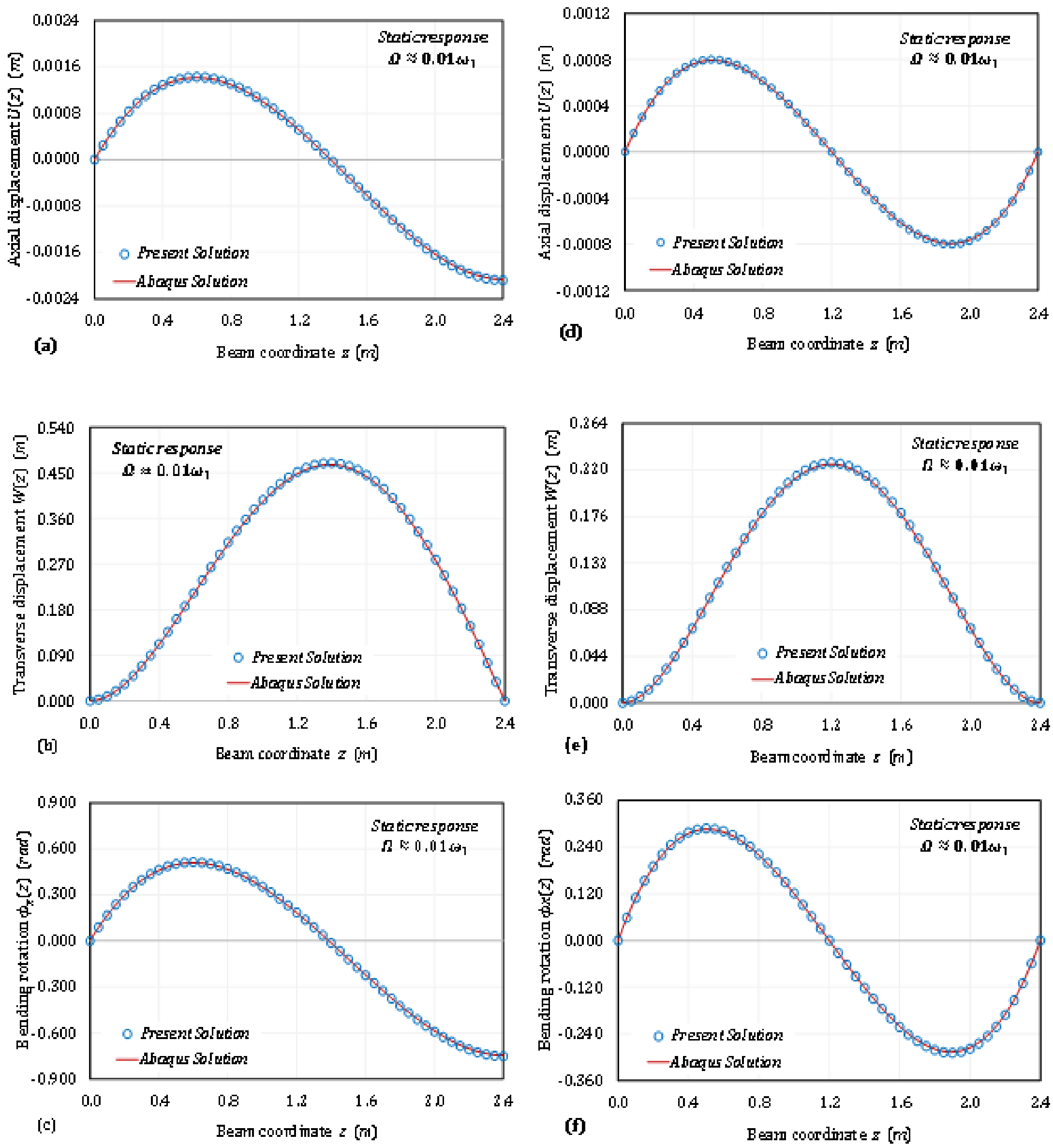

Figure (6): Static Responses for Asymmetric Cross-Ply (0º/90º/0º/90º) Laminated Clamped-Roller and Clamped-Clamped Beams Under Distributed Transverse Harmonic Force.

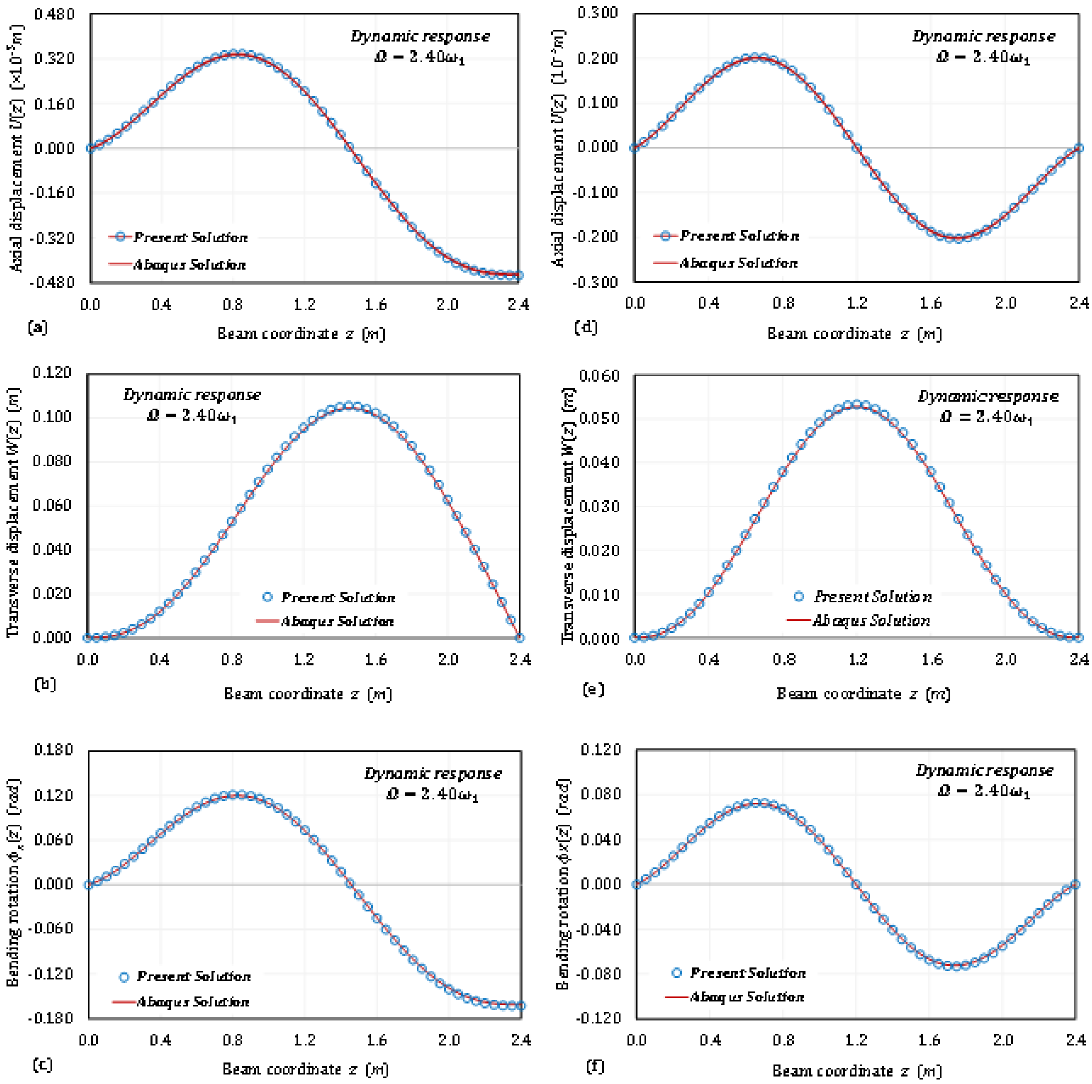

Figure (7): Dynamic Responses for Asymmetric Cross-Ply (0º/90º/0º/90º) Laminated Clamped-Roller and Clamped-Clamped Beams Under Distributed Harmonic Force.

Conclusions

Based on the variational form of Hamilton's principle, the governing dynamic equations and associated boundary conditions were derived for coupled extensional-flexural response of asymmetric laminated beams with rectangular cross-sections subjected to various harmonic bending forces. The analytical closed-form solutions of extensional-flexural coupled equations were obtained for composite asymmetric laminated beams with cantilevered, simply supported, clamped-clamped and clamped-roller boundary conditions.

Comparing the present results with those based on established Abaqus finite element and exact solutions available in the literature demonstrate the validity and accuracy of the present closed-form solutions. Hence, it can be concluded that the present analytical solutions successfully captured the quasi-static and steady state dynamic responses for composite asymmetric laminated beams under different harmonic bending forces. Moreover, the closed-from solutions are capable of extracting the coupled natural frequencies and steady state modes.

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